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Entropy solutions for some nonlinear elliptic unilateral
problems in anisotropic Sobolev spaces

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Abstract

In this paper, we consider the following nonlinear elliptic unilateral equations of the type

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + g(x, u, \nabla u) + |u|^{p_0-2} u = f \text{ in } \Omega$$

In the anisotropic Sobolev space, we prove the existence of entropy solutions for our unilateral problem, the function $g(x, u, \nabla u)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum f belongs to $L^1(\Omega)$.

key words : Anisotropic Sobolev spaces, nonlinear elliptic unilateral problem, entropy solutions.

1- Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Boccardo and Gallouët have considered in [14] the elliptic problem

$$\begin{cases} -\operatorname{div}(x, u, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where f is a bounded Radon measure on Ω . They have proved the existence of solutions $u \in W_0^{1,q}(\Omega)$ for all $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$. Also, they have proved some regularity results. In [11], Boccardo has studied the existence of entropy solutions for the quasilinear elliptic problem of the type

$$\begin{cases} -\operatorname{div}(x, u, \nabla u) = f - \operatorname{div}\phi(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

with $f \in L^1(\Omega)$ and $\phi(u) \in C^0(\mathbb{R}, \mathbb{R}^N)$. He has proved the existence and some regularity of solutions. The most studied problems were involved in the p -Laplace operators

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

with p constant, where the authors have proved the existence of solutions for some nonlinear elliptic unilateral problems in Sobolev spaces. (see for example [13, 14, 20]).

In the framework of Orlicz Sobolev spaces, Aharouch and Bennouna [1] have treated the quasilinear elliptic unilateral problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solutions $u \in W_0^1 L_M(\Omega)$ without any restriction on the N -function M of the Orlicz spaces (i.e. without assuming Δ_2 -condition), we refer also to [5, 10, 16] for more details.





Recently, the anisotropic Sobolev spaces have attracted the attention of many scientists and researchers (see. [7, 18, 22]), this impulse mainly comes from their physical applications in the processes of image restoration, flows of electro-rheological fluids and thermistor problem.

M. AL-hawmi, E. Azroul, H. Hjiat and A. Touzani have studied in [2] the existence of entropy solutions for some anisotropic quasilinear elliptic unilateral problems

$$\begin{cases} Au = \mu - \operatorname{div} \phi(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.4)$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}(\Omega)$ into its dual define by

$$Au = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$

where $\phi(u) \in C^0(\mathbb{R}, \mathbb{R}^N)$ and $\mu = f - \operatorname{div} F$ such that $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'_i}(\Omega)$.

L. M. Kozhevnikova has proved that the entropy solution obtained is a normalized solution in [19] of the problem under consideration.

$$\begin{cases} \sum_{i=1}^N (a_i(x, u, \nabla u))_{x_i} = |u|^{p(x)-2}u + b(x, u, \nabla u) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

AL-hawmi, A. Benkirane, H. Hjiat and A. Touzani have studied in [3] the existence and uniqueness of entropy solutions for some nonlinear elliptic unilateral problems

$$\begin{cases} Au = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.6)$$

in Musielak-Orlicz-Sobolev spaces, where $f \in L^1(\Omega)$ and

$A : D(A) \subset W_0^1 L_\varphi(\Omega) \rightarrow W_0^1 L_\psi(\Omega)$ is the Leray-Lions operator defined as:

$$Au = -\operatorname{div} a(x, \nabla u)$$

In this paper, we will prove the existence of entropy solutions for nonlinear anisotropic unilateral elliptic problem of the type

$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{p_0-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.7)$$

where Au is an operator of Leray-Lions type acting from $W_0^{1,\vec{p}}(\Omega)$ into its dual define by

$$Au = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$





and the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ satisfying the sign condition, that is $g(x, u, \nabla u) u \geq 0$.

This paper is organized as follows. In section 2 we recall some definitions and basic properties concerning the anisotropic Sobolev spaces. We introduce in section 3 the assumptions on $a_i(x, u, \nabla u)$ and $g(x, u, \nabla u)$ for which our problem has at least one solution. The section 4 will be devoted to show the existence of entropy solutions for our anisotropic nonlinear elliptic unilateral problem (1.7).

2- Preliminary

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$ with boundary $\partial\Omega$. Let $p_0, p_1, \dots, p_N$ be $N + 1$ exponents, with $1 < p_i < \infty$ for $i = 1, \dots, N$. We denote

$$\vec{p} = (p_0, p_1, \dots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N$$

We set

$$\underline{p} = \min \{p_0, p_1, \dots, p_N\} \quad \text{then} \quad \underline{p} > 1. \quad (2.1)$$

The anisotropic Sobolev space $W^{1, \vec{p}}(\Omega)$ is defined as follows

$W^{1, \vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \text{ and } D^i u \in L^{p_i}(\Omega) \text{ for } i = 1, 2, \dots, N\}$ endowed with the norm

$$\|u\|_{1, \vec{p}} = \sum_{i=1}^N \|D^i u\|_{L^{p_i}(\Omega)}. \quad (2.2)$$

The space $(W^{1, \vec{p}}(\Omega), \|u\|_{1, \vec{p}})$ is a separable and reflexive Banach space (cf [22]).

We define also $W_0^{1, \vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1, \vec{p}}(\Omega)$ with respect to the norm (2.2).

Now, we will present some important compacts embedding,

Lemma 2.1 Let Ω be a bounded open set in $\mathbb{R}^N$, then the following embedding are compact

- if $\underline{p} < N$ then $W_0^{1, \vec{p}}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \underline{p}^*[,$
where $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N},$
- if $\underline{p} = N$ then $W_0^{1, \vec{p}}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \infty[,$
- if $\underline{p} > N$ then $W_0^{1, \vec{p}}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^0(\bar{\Omega}).$

The proof of this lemma follows from the fact that the embedding $W_0^{1, \vec{p}}(\Omega) \hookrightarrow W_0^{1, \underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Definition 2.1 Let $k > 0$, we consider the truncation function $T_k(\cdot): \mathbb{R} \rightarrow \mathbb{R}$ given by





$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| \geq k, \end{cases}$$

and we define

$$T_0^{1,\vec{p}}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \text{ measurable, } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}$$

Proposition 2.1 $u \in T_0^{1,\vec{p}}(\Omega)$, for $i = 1, \dots, N$, exists a unique measurable function $v_i : \Omega \rightarrow \mathbb{R}$ such that

$$D^i T_k(u) = v_i x_{\{|u| < k\}} \text{ a.e. in } \Omega, \text{ for any } k > 0,$$

where x_A denotes the characteristic function of a measurable set A . The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivatives of u , that is $v_i = D^i u$.

The proof follows the usual techniques developed in [12] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8] and [15].

3 - Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), we assume that the vector $\vec{p} = (p_0, p_1, \dots, p_N)$ satisfying the requirements that $1 < p_i < \infty$ for $i = 1, 2, \dots, N$.

Taking ψ as a measurable function on Ω with values in $\mathbb{R}$ such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

We define the convex subset K_ψ by

$$K_\psi = \{v \in W_0^{1,\vec{p}}(\Omega) \text{ such that } v \geq \psi \text{ a.e. in } \Omega\}.$$

We consider a Leray-Lions operator $A : W_0^{1,\vec{p}}(\Omega) \rightarrow W^{-1,\vec{p}'}(\Omega)$ given by

$$Au = - \sum_{i=1}^N D^i a_i(x, u, \nabla u)$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ are Carathéodory functions, for $i = 1, 2, \dots, N$, (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfy the following conditions

$$|a_i(x, s, \xi)| \leq \beta(R_i(x) + |s|^{p_i-1} + |\xi|^{p_i-1}) \text{ for } i = 1, 2, \dots, N, \quad (3.1)$$

$$a_i(x, s, \xi) \xi_i \geq \alpha |\xi_i|^{p_i} \text{ for } i = 1, 2, \dots, N, \quad (3.2)$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi_i))(\xi_i - \xi_i) > 0 \text{ for } \xi_i \neq \xi_i \quad (3.3)$$

for a.e. x in Ω and all (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, where $R_i(x)$ is a nonnegative function lying in $L^{p_i'}(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $g(x, s, \xi)$ is a Carathéodory function which satisfies

$$g(x, s, \xi) s \geq 0 \quad (3.4)$$

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + \sum_{i=1}^N |\xi_i|^{p_i}) \quad (3.5)$$

where $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous nondecreasing function, $C \in L^1(\Omega)$. We consider the anisotropic nonlinear elliptic unilateral problem





$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{p_0-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3.6)$$

Where

$$f \in L^1(\Omega) \quad (3.7)$$

Now, we recall some important Lemmas useful to prove our main result.

Lemma 3.1 (see [17], Theorem 13.47 page 216) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that

- (i) $u_n \rightarrow u$ a.e. in Ω ,
- (ii) $u_n \geq 0$ and $u \geq 0$ a.e. in Ω ,
- (iii) $\int_{\Omega} u_n dx \rightarrow \int_{\Omega} u dx$,

then $u_n \rightarrow u$ in $L^1(\Omega)$.

Lemma 3.2 (see. [7] page 6) Assuming that (3.1) - (3.3) hold, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightarrow u$ in $W_0^{1,\vec{p}}(\Omega)$ and

$$\begin{aligned} \int_{\Omega} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u)dx \\ + \sum_{i=1}^N \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D^i u_n - D^i u)dx \rightarrow 0, \end{aligned} \quad (3.8)$$

then $u_n \rightarrow u$ in $W_0^{1,\vec{p}}(\Omega)$ for a subsequence.

4- Main results.

We first give the definition of entropy solutions for our problem (3.6) as follows.

Definition 4.1 A measurable function u is called entropy solution of the anisotropic nonlinear elliptic unilateral problem (3.6) if $T_k(u) \in K_{\psi}$ and satisfy

$$\left\{ \begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) D^i T_k(u_n - v) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u_n - v) dx \\ \int_{\Omega} |u|^{p_0-2} u T_k(u_n - v) dx \leq \int_{\Omega} f T_k(u_n - v) dx \\ \text{for any } v \in W_0^{1,\vec{p}}(\Omega) \cap L^{\infty}(\Omega) \end{aligned} \right. \quad (4.1)$$

Our objective is to prove the following existence theorem.

Theorem 4.1 Assuming that (3.1) - (3.5) and (3.7) hold, then the problem (3.6) has at least one entropy solution.

Proof of Theorem 4.1

Step 1: Approximate problems.





Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $W^{-1, \vec{p}'}(\Omega) \cap L^1(\Omega)$ such that $f_n \rightarrow f$ in $L^1(\Omega)$ and

$|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem.

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i(u_n - v) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n)(u_n - v) dx \\ & + \int_{\Omega} |u_n|^{p_0-2} u_n(u_n - v) dx \leq \int_{\Omega} f_n(u_n - v) dx \end{aligned} \quad (4.2)$$

for any $v \in W_0^{1, \vec{p}}(\Omega) \cap L^{\infty}(\Omega)$

for any $v \in K_{\psi}$, and $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}$, $g_n(x, s, \xi)s \geq 0$,

$|g_n(x, s, \xi)| \leq |g(x, s, \xi)|$ and $|g_n(x, s, \xi)| \leq n$. We define the operators A_n and G_n acted from $W_0^{1, \vec{p}}(\Omega)$ into its dual $W^{-1, \vec{p}'}(\Omega)$ by

$$\begin{aligned} \langle A_n u, v \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i v dx + \int_{\Omega} |u|^{p_0-2} u v dx \\ &\quad \forall u, v \in W_0^{1, \vec{p}}(\Omega) \end{aligned}$$

and

$$\langle G_n u, v \rangle = \int_{\Omega} g(x, u, \nabla u) v dx \quad \forall u, v \in W_0^{1, \vec{p}}(\Omega).$$

Lemma 4.1 The operator $B_n = A_n + G_n$ from $W_0^{1, \vec{p}}(\Omega)$ into $W^{-1, \vec{p}'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense: there exists $v_0 \in K_{\psi}$ such that

$$\frac{\langle B_n u, v - v_0 \rangle}{\|u\|_{1, \vec{p}}} \rightarrow \infty \quad \text{as } \|u\|_{1, \vec{p}} \rightarrow \infty \quad \text{for } v \in K_{\psi}.$$

In view of **Lemma 5.1** (see **Appendix**), there exists at least one solution $u_n \in W_0^{1, \vec{p}}(\Omega)$ of the anisotropic nonlinear elliptic unilateral problem (4.2) (cf. [21], **Theorem 8.2 page131**).

Step 2: A priori estimates.

Let $k \geq \max(1, \|\psi^+\|_{\infty})$; we set $v = u_n - \eta T_k(u_n - \psi^+) W_0^{1, \vec{p}}(\Omega) \cap L^{\infty}(\Omega)$ for η small enough we have $v \in K_{\psi}$, thus v is an admissible test function in (4.2), and we obtain

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \psi^+) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - \psi^+) dx \\ & + \int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \leq \int_{\Omega} f_n T_k(u_n - \psi^+) dx \end{aligned} \quad (4.3)$$



From (4.3), it follows that

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx + \int_{\{|u_n - \psi^+| \leq k\}} g_n(x, u_n, \nabla u_n) u_n dx \\ & + \int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \leq \int_{\Omega} f_n T_k(u_n - \psi^+) dx \quad (4.4) \\ & + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i \psi^+ dx + \int_{\{|u_n - \psi^+| \leq k\}} g_n(x, u_n, \nabla u_n) \psi^+ dx \end{aligned}$$

Since $k \geq \|\psi^+\|_{\infty}$ then $T_k(u_n - \psi^+)$ have the same sign as u_n on the set $\{|u_n - \psi^+| > k\}$ it follows that

$$\begin{aligned} & \int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \\ & = \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-2} u_n (u_n - \psi^+) dx \\ & + \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \\ & \geq \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0} - \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| dx \end{aligned}$$

In view of (3,2), we obtain

$$\begin{aligned} & \propto \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} D^i u_n + \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0} \\ & \leq \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx \\ & + \int_{\{|u_n - \psi^+| \leq k\}} g_n(x, u_n, \nabla u_n) u_n dx \\ & \leq \int_{\Omega} f_n T_k(u_n - \psi^+) dx \quad (4.5) \\ & + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)| |D^i \psi^+| dx \\ & + \int_{\{|u_n - \psi^+| \leq k\}} |g_n(x, u_n, \nabla u_n)| |\psi^+| dx \\ & + \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| dx \end{aligned}$$

Thanks to Young's inequality we have





$$\int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| dx \leq \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} |\psi^+| dx + C_1 \quad (4.6)$$

also, in view of (3,1) and Young's inequality, we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)| |D^i \psi^+| dx \\ & \leq \beta \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} R_i(x) |D^i \psi^+| dx \\ & + \beta \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p-1} |D^i \psi^+| dx \\ & + \beta \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_{i-1}} |D^i \psi^+| dx \leq C_2 \\ & + C_3 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^p dx + \frac{\alpha}{4} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_i} dx \\ & + C_4 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i \psi^+|^p dx + C_5 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i \psi^+|^{p_i} dx, \end{aligned} \quad (4.7)$$

also, in view of (3.5) and Young's inequality, we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |g_n(x, u_n, \nabla u_n)| |D^i \psi^+| dx \\ & \leq \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} b|u_n| (c(x) + |D^i u_n|^{p_{i-1}}) |D^i \psi^+| dx \\ & \leq C_6 + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_{i-1}} |D^i \psi^+| dx \\ & \leq C_6 + \frac{\alpha}{2} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_i} dx \\ & + C_7 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i \psi^+|^{p_i} dx, \end{aligned} \quad (4.8)$$

By combining (4.5) - (4.8), we deduce that

$$\begin{aligned} & \frac{\alpha}{4} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_i} dx + \frac{1}{2} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} dx \\ & \leq k \|f\|_1 + N \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^p dx + C_8 \end{aligned} \quad (4.9)$$





Then, there exists a constant C_9 depending only on k such that

$$\frac{\alpha}{4} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_i} dx \leq C_9(k) \quad (4.10)$$

and since

$$\{x \in \Omega, |u_n| \leq k\} \subset \{x \in \Omega, |u_n - \psi^+| \leq k + \|\psi^+\|_\infty\}$$

Therefore

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i} dx &= \sum_{i=1}^N \int_{\{|u_n| \leq k\}} |D^i u_n|^{p_i} dx + \int_{\Omega} |T_k(u_n)|^{p_0} dx \\ &\leq \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k + \|\psi^+\|_\infty\}} |D^i u_n|^{p_i} dx + |k|^{p_0} |\Omega| \\ &\leq C_{10}(k, \|\psi^+\|_\infty), \end{aligned}$$

and we obtain

$$\|T_k(u_n)\|_{1, \bar{p}} \leq C_{11}(k, \|\psi^+\|_\infty),$$

where C_{11} is positive constant that does not depend on n . Thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1, \bar{p}}(\Omega)$ uniformly in n , then there exists a subsequence still denoted $(T_k(u_n))_{n \in N}$ and a function $v_k \in W_0^{1, \bar{p}}(\Omega)$

such that

$$\left\{ \begin{array}{l} T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^{1, \bar{p}}(\Omega), \\ T_k(u_n) \rightarrow v_k \text{ strongly in } L^{\bar{p}}(\Omega) \text{ and a.e in } \Omega \end{array} \right\} \quad (4.11)$$

On the other hand, thanks to (4.9) and Poincaré inequality, we have

$$\begin{aligned} \|\nabla T_k(u_n)\|_{\bar{p}}^{\bar{p}} &= \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{\bar{p}} dx \leq \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i} dx + N|\Omega| \\ &\leq \frac{4}{\alpha} k \|f\|_1 + \frac{4}{\alpha} N \|T_k(u_n)\|_{\bar{p}}^{\bar{p}} + C_{12} \\ &\leq \frac{4}{\alpha} k \|f\|_1 + \frac{4}{\alpha} CN \|\nabla T_k(u_n)\|_{\bar{p}}^{\bar{p}} + C_{12} \end{aligned} \quad (4.12)$$

Therefore, by taking $\frac{4}{\alpha} CN \leq \frac{1}{3}$, there exists a constant C_{13} that does not depend on k and n , such that

$$\|\nabla T_k(u_n)\|_{\bar{p}} \leq C_{13} k^{\frac{1}{3}} \quad \text{for} \quad k \geq 1$$

and we obtain

$$\begin{aligned} k \text{ meas } \{|u_n| > k\} &= \int_{\{|u_n| > k\}} |T_k(u_n)| dx \leq \int_{\Omega} |T_k(u_n)| dx \\ &\leq \|1\|_{\bar{p}'} \|T_k(u_n)\|_{\bar{p}} \end{aligned} \quad (4.13)$$





$$\leq C \|\nabla T_k(u_n)\|_p \leq C_{14} k^{\frac{1}{p}},$$

which yields

$$\text{meas} \{|u_n| > k\} \leq C_{14} \frac{1}{k^{\frac{1}{1-\frac{1}{p}}}} \rightarrow 0, \quad k \rightarrow \infty \quad (4.14)$$

Now, we will show that the sequence $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$,

$$\begin{aligned} \text{meas} \{|u_n - u_m| > \delta\} \\ \leq \text{meas} \{|u_n| > k\} + \text{meas} \{|u_m| > k\} \\ + \text{meas} \{|T_k(u_n) - T_k(u_m)| > \delta\} \end{aligned}$$

Let $\varepsilon > 0$, in view of (4.14) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas} \{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas} \{|u_m| > k\} \leq \frac{\varepsilon}{3} \quad (4.15)$$

Moreover, thanks to (4.11) we have

$$T_k(u_n) \rightarrow \eta_k \text{ strongly in } L^p(\Omega) \text{ and a.e in } \Omega$$

Thus $(T_k(u_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas} \{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \text{ for all } m, n \geq n_0(k, \delta, \varepsilon) \quad (4.16)$$

By combining (4.15) and (4.16), we conclude that : for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas} \{|u_n - u_m| > \delta\} \leq \varepsilon \quad \text{for any } m, n \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u . Thanks to (4.11) we obtain

$$\left\{ \begin{array}{ll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega), \\ T_k(u_n) \rightarrow T_k(u) & \text{strongly in } L^p(\Omega) \text{ and a.e in } \Omega \end{array} \right\} \quad (4.17)$$

Step 3: Convergence of the gradient.

In the sequel, we denote by $\varepsilon_i(n)$ $i = 1, 2 \dots$ a various functions of real numbers which converges to 0 as n tends to infinity.

Let $\varphi_k(s) = s \cdot e^{\gamma s^2}$ where $\gamma = \left(\frac{b(k)}{2\alpha}\right)^2$. It is obvious that

$$\varphi'_k(s) - \frac{b(k)}{\alpha} |\varphi_k(s)| \geq \frac{1}{2} \quad \forall s \in \mathbb{R}.$$

We also consider $h > k > 0$ and $M = 4k + h$ and we set

$$\omega_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n)) - T_k(u).$$

By taking $v = u_n - \eta \varphi_k(\omega_n)$ we have $v \geq \psi$ for η small enough, thus v is an admissible test function in (4.2), and we obtain





$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) \varphi'_k(\omega_n) D^i \omega_n dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \\ + \int_{\Omega} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx \leq \int_{\Omega} f_n \varphi_k(\omega_n) dx$$

It is easy to see that $D^i \omega_n = 0$ on $\{|u_n| > M\}$ and since

$g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n) \geq 0$ on $\{|u_n| > k\}$, then

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \\ + \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx \leq \int_{\Omega} f_n \varphi_k(\omega_n) dx \quad (4.18)$$

First estimate :

We have

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\ = \sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i T_{2k}(u_n - T_k(u)) dx \quad (4.19) \\ + \sum_{i=1}^N \int_{\{|u_n| > k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n$$

On one hand, since $|u_n - T_k(u)| \leq 2k$ on $\{|u_n| \leq k\}$, then

$$\sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) D^i T_{2k}(u_n - T_k(u)) dx \\ = \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx \quad (4.20) \\ + \sum_{i=1}^N \int_{\{|u_n| > k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) D^i T_k(u) dx$$

Since $1 \leq \varphi'_k(\omega_n) \leq \varphi'_k(2k)$, then

$$\left| \sum_{i=1}^N \int_{\{|u_n| > k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) D^i T_k(u) dx \right| \\ \leq \varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n| > k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx$$





and since $(|a_i(x, T_k(u_n), \nabla T_k(u_n))|)_{n \in \mathbb{N}}$ is bounded in $L^{p'_i}(\Omega)$, then there exists $\vartheta \in L^{p'_i}(\Omega)$ such that $(|a_i(x, T_k(u_n), \nabla T_k(u_n))|) \rightarrow \vartheta$ in $L^{p'_i}(\Omega)$. Therefore

$$\begin{aligned} \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \\ \rightarrow \sum_{i=1}^N \int_{\{|u_n|>k\}} \vartheta |D^i T_k(u)| dx = 0 \end{aligned}$$

It follows that

$$\sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i T_k(u) dx = \varepsilon_0(n) \quad (4.21)$$

On the other hand, for the second term on the right hand side of (4.19), taking $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, then

$$\begin{aligned} \sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\ = \sum_{i=1}^N \int_{\{|u_n|>k\} \cap \{|z_n| \leq 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i z_n dx \\ + \sum_{i=1}^N \int_{\{|u_n|>k\} \cap \{|z_n| \leq 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i (u_n - T_k(u)) \chi_{\{|u_n|>h\}} dx \\ - \sum_{i=1}^N \int_{\{|u_n|>k\} \cap \{|z_n| \leq 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i T_k(u) \chi_{\{|u_n| \leq h\}} dx \\ \geq -\varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \end{aligned}$$

In the same way as for (4.21), we can prove that

$$\varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx = \varepsilon_1(n). \quad (4.22)$$

After adding (4.19) - (4.22), we obtain

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\ \geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx + \varepsilon_1(n) \end{aligned} \quad (4.23)$$

which is equivalent to say



$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx \\
 & \leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\
 & - \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx - \varepsilon_2(n) \\
 & \leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\
 & + \varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx - \varepsilon_2(n)
 \end{aligned}$$

Concerning the second term on the right-hand side of (4.24), by applying the Lebesgue convergence theorem, we have $T_k(u_n) \rightarrow T_k(u)$ in $L^{p_0}(\Omega)$, then, $|a_i(x, T_k(u_n), \nabla T_k(u))| \rightarrow |a_i(x, T_k(u), \nabla T_k(u))|$ in $L^{p_i}(\Omega)$, and since $D^i T_k(u_n)$ converges to $D^i T_k(u)$ weakly in $L^{p_i}(\Omega)$, we obtain

$$\varepsilon_3(n) = \varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \rightarrow 0$$

as $n \rightarrow \infty$ (4.25)

We conclude that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) \varphi'_k(\omega_n) (D^i T_k(u_n) \\
 & - D^i T_k(u)) dx \\
 & \leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx + \varepsilon_3(n)
 \end{aligned}$$

(4.26)

Second estimate :

Now, we treat the second term on the left-hand side of (4.18). From (3.5), we obtain

$$\begin{aligned}
 & \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right| \\
 & \leq \int_{\{|u_n| \leq k\}} b(|u_n|) \left(c(x) + \sum_{i=1}^N |D^i T_k(u_n)|^{p_i} \right) |\varphi_k(\omega_n)| dx \\
 & \leq b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx
 \end{aligned}$$





$$\begin{aligned}
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u_n) |\varphi_k(\omega_n)| dx \\
 & \leq b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx \\
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| dx \\
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
 & \quad (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & \text{therefore,} \\
 & \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
 & \quad (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & \geq \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right| - b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx \\
 & - \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & - \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| dx. \tag{4.27}
 \end{aligned}$$

We have $\varphi_k(\omega_n) \rightarrow \varphi_k(T_{2k}(u - T_h(u)))$ weak-* in $L^\infty(\Omega)$ as $n \rightarrow \infty$ as, the

$$\int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx \rightarrow \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(T_{2k}(u - T_h(u)))| dx = 0 \tag{4.28}$$

Concerning the third term on the right-hand side of (4.27), thanks to (4.25), we have

$$\begin{aligned}
 & \left| \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \right| \\
 & \leq \varphi_k(2k) \sum_{i=1}^N \int_{\Omega} |a_i(x, T_k(u_n), \nabla T_k(u))| |D^i T_k(u_n) - D^i T_k(u)| dx \\
 & \rightarrow 0 \text{ as } n \rightarrow \infty \tag{4.29}
 \end{aligned}$$

for the last term on the right-hand side of (4.27), we have that

$a_i(x, T_k(u_n), \nabla T_k(u_n))_{n \in \mathbb{N}}$ is bounded in $L^{p_i}(\Omega)$, then there exists

$\psi_i \in L^{p_i}(\Omega)$ such that $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \psi_i$ in $L^{p_i}(\Omega)$ and, using the





fact that

$$D^i T_k(u) |\varphi_k(\omega_n)| \rightarrow D^i T_k(u) \varphi_k(T_{2k}(u - T_h(u))) \quad \text{in } L^{p_i}(\Omega)$$

it follows that

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| dx \\ \rightarrow \sum_{i=1}^N \int_{\Omega} \psi_i D^i T_k(u) |\varphi_k(T_{2k}(u - T_h(u)))| dx = 0 \end{aligned} \quad (4.30)$$

By combining (4.27) - (4.30), we get

$$\begin{aligned} \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\ \geq \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right| + \varepsilon_5(n) \end{aligned} \quad (4.31)$$

Third estimate :

We have

$$\begin{aligned} \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx \\ = \int_{\Omega} |T_k(u_n)|^{p_0-2} T_k(u_n) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ - \int_{\{|u_n| > k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ = \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) \\ - T_k(u)) e^{\gamma \omega_n^2} dx \\ + \int_{\Omega} |T_k(u)|^{p_0-2} T_k(u) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ - \int_{\{|u_n| > k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ \geq \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) \\ - T_k(u)) dx \\ - e^{\gamma (2k)^2} \int_{\Omega} |T_k(u)|^{p_0-1} |T_k(u_n) - T_k(u)| dx \\ - e^{\gamma (2k)^2} \int_{\{|u_n| > k\}} k^{p_0-1} |T_k(u_n) - T_k(u)| dx \end{aligned}$$

and as $T_k(u_n) \rightarrow T_k(u)$ in $L^{p_0}(\Omega)$, then the second and the last term on the right-hand side of the inequality above converges to 0 as n goes to infinity. Therefore, we get





$$\int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) - T_k(u)) dx$$

$$\leq \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n (T_k(u_n) - T_k(u)) \varphi_k(\omega_n) dx + \varepsilon_6(n) \quad (4.32)$$

Thanks to (4.26), (4.31) and (4.32), we obtain

$$\frac{1}{2} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (D^i T_k(u_n) - D^i T_k(u)) \varphi_k(\omega_n) dx$$

$$+ \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) - T_k(u)) dx$$

$$\leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi_k'(\omega_n) D^i \omega_n dx$$

$$- \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right|$$

$$+ \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx + \varepsilon_7(n)$$

$$\leq \int_{\Omega} f_n \varphi_k(\omega_n) dx + \varepsilon_7(n) \quad (4.33)$$

We have

$$\int_{\Omega} f_n \varphi_k(\omega_n) dx = \int_{\Omega} f_n \varphi_k(u - T_h(u)) dx + \varepsilon_8(n) \quad (4.34)$$

By combining (4.33) and (4.34), we obtain

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (D^i T_k(u_n) - D^i T_k(u)) dx$$

$$+ \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) - T_k(u)) dx$$

$$\leq \int_{\Omega} f_n \varphi_k(u - T_h(u)) dx + \varepsilon_9(n) \quad (4.35)$$

Then, by letting h and n goes to infinity in (4.35), we get

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (D^i T_k(u_n) - D^i T_k(u)) dx$$

$$+ \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) - T_k(u)) dx \rightarrow 0$$

$$(4.36)$$

Using the **Lemma 3.2**, we deduce that

$$T_k(u_n) \rightarrow T_k(u) \text{ in } W_0^{1,\vec{p}}(\Omega) \quad (4.37)$$

Therefore,

$$D^i T_k(u_n) \rightarrow D^i T_k(u) \text{ a. e. in } \Omega$$



Step 4: The equi-integrability of $g(x, u_n, \nabla u_n)$ and $|u_n|^{p_0-2} u_n$.

In order to pass to the limit in the approximate equation, we show that

$$g(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ and } |u_n|^{p_0-2} u_n \rightarrow |u|^{p_0-2} u \text{ strongly in } L^1(\Omega)$$

By using Vitali's Theorem, it suffices to prove that $g(x, u_n, \nabla u_n)$ and $|u_n|^{p_0-2} u_n$ are uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (4.2), and using (3,2) since $T_1(u_n - T_h(u_n))$ have the same sign as u_n , we obtain

$$\begin{aligned} & \int_{\{h > |u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n)) dx \\ & + \int_{\{h > |u_n|\}} |u_n|^{p_0-2} u_n T_1(u_n - T_h(u_n)) dx \\ & \leq \int_{\{h > |u_n|\}} f_n T_1(u_n - T_h(u_n)) dx \end{aligned} \quad (4.38)$$

It follows that

$$\begin{aligned} & \int_{\{h+1 \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| dx \\ & + \int_{\{h+1 \leq |u_n|\}} |u_n|^{p_0-1} dx \\ & \leq \int_{\{h < |u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n)) dx \\ & + \int_{\{h < |u_n|\}} |u_n|^{p_0-2} u_n T_1(u_n - T_h(u_n)) dx \\ & \leq \int_{\{h < |u_n|\}} f_n T_1(u_n - T_h(u_n)) dx \leq \int_{\{h < |u_n|\}} |f| dx \end{aligned}$$

thus, for all $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\int_{\{h(\eta) \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{\{h(\eta) \leq |u_n|\}} |u_n|^{p_0-1} dx \leq \frac{\eta}{2} \quad (4.39)$$

On the other hand, for any measurable subset $E \subset \Omega$, we have





$$\begin{aligned}
 & \int_E |g_n(x, u_n, \nabla u_n)| dx + \int_E |u_n|^{p_0-1} dx \\
 & \leq b(h(\eta)) \int_E \left(c(x) + \sum_{i=1}^N |D^i T_{h(\eta)}(u_n)|^{p_i} \right) dx \\
 & + \int_E |T_{h(\eta)}(u_n)|^{p_0-1} dx + \int_{\{h(\eta) \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| dx \\
 & + \int_{\{h(\eta) \leq |u_n|\}} |u_n|^{p_0-1} dx
 \end{aligned} \quad (4.40)$$

From (4.37), there exists $\beta(\eta) > 0$ such that, for all $E \subseteq \Omega$, with $meas(E) \leq \beta(\eta)$, we have

$$b(h(\eta)) \int_E \left(c(x) + \sum_{i=1}^N |D^i T_{h(\eta)}(u_n)|^{p_i} \right) dx + \int_E |T_{h(\eta)}(u_n)|^{p_0-1} dx \leq \frac{\eta}{2} \quad (4.41)$$

Finally, by combining (4.39), (4.40) and (4.41), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)| dx + \int_E |u_n|^{p_0-1} dx \leq \eta \quad (4.42)$$

for all E such that $meas(E) \leq \beta(\eta)$,

we then deduce that $(g_n(x, u_n, \nabla u_n))_n$ and $(|u_n|^{p_0-2} u_n)_n$ are equi-integrable, and by Vitali's Theorem we deduce that

$$g(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ and } |u_n|^{p_0-2} u_n \rightarrow |u|^{p_0-2} u \text{ in } L^1(\Omega) \quad (4.43)$$

Step 5: Passing to the limit.

Let $v \in K_\psi \cap L^\infty(\Omega)$, by taking $v = u_n - \eta T_k(u_n - \varphi)$ as a test function in (4.2), with η small enough and $\varphi \in K_\psi \cap W_0^{1,\vec{p}}(\Omega)$, and putting $M = k + \|\varphi\|_\infty$, we get obtain

$$\begin{aligned}
 & \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\
 & + \int_\Omega |u_n|^{p_0-2} u_n T_k(u_n - \varphi) dx \leq \int_\Omega f_n T_k(u_n - \varphi) dx
 \end{aligned} \quad (4.44)$$

On one hand, if $|u_n| > M$ then $|u_n - \varphi| \geq |u_n| - |\varphi| > k$, therefore $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$, which implies that



$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx \\
 & + \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & = \sum_{i=1}^N \int_{\Omega} (a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla \varphi)) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & + \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla \varphi) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \quad (4.45)
 \end{aligned}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx \\
 & \geq \sum_{i=1}^N \int_{\Omega} (a_i(x, T_M(u_n), \nabla T_M(u_n)) \\
 & \quad - a_i(x, T_M(u_n), \nabla \varphi)) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & + \lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla \varphi) (D^i T_M(u_n) \\
 & \quad - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \quad (4.46)
 \end{aligned}$$

The second term in the right hand side of (4.46) is equal to

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla \varphi) (D^i T_M(u) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx$$

Therefore, we get

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx \\
 & \geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u), \nabla \varphi) (D^i T_M(u) - D^i \varphi) \chi_{\{|u - \varphi| \leq k\}} dx \\
 & = \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) (D^i u - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & = \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) D^i T_k(u - \varphi) dx
 \end{aligned}$$

On the other hand, we have $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$ weak-* in $L^\infty(\Omega)$, and in view of (4.43), We obtain,





$$\int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \rightarrow \int_{\Omega} g_n(x, u, \nabla u) T_k(u - \varphi) dx, \quad (4.47)$$

$$\int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \varphi) dx \rightarrow \int_{\Omega} |u|^{p_0-2} u T_k(u - \varphi) dx \quad (4.48)$$

$$\text{and } \int_{\Omega} f_n T_k(u_n - \varphi) dx \rightarrow \int_{\Omega} f T_k(u - \varphi) dx \quad (4.49)$$

Again, since $T_k(u_n - \varphi) \rightarrow T_k(u - \varphi)$ in $W_0^{1,\vec{p}}(\Omega)$. Which completes the proof of Theorem 4.1. for any positive function $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

5- Appendix

Lemma 5.1 The bounded operator $B_n = A_n + G_n$ acted from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense : there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1,\vec{p}}} \rightarrow \infty \quad \text{as } \|v\|_{1,\vec{p}} \rightarrow \infty \quad \text{for } v \in K_\psi.$$

Proof of Lemma 5.1

Using the *Hölder's* inequality and the growth condition (3.1), we can show that the operator A_n is bounded and by (5.2) we conclude that B_n is bounded. For the coercivity, let $v_0 \in K_\psi$, for any $v \in K_\psi$ we have

$$\begin{aligned} \langle A_n v, v_0 \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(v), \nabla v) D^i v_0 dx + \int_{\Omega} |v|^{p_0-2} v v_0 dx \\ &\leq \sum_{i=1}^N \left(\int_{\Omega} |a_i(x, T_n(v), \nabla v)|^{p'_i} dx \right)^{\frac{1}{p'_i}} \|D^i v_0\|_{p_i} \\ &\quad + \left(\int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \|v_0\|_{p_0} \\ &\leq c_{15} \sum_{i=1}^N \left(\int_{\Omega} (R_i(x)^{p'_i} + n^{(p-1)p'_i} + |D^i v|^{p_i}) dx \right)^{\frac{1}{p'_i}} \|v_0\|_{1,\vec{p}} \\ &\quad + \left(\int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \|v_0\|_{1,\vec{p}} \end{aligned}$$

and

$$\langle A_n v, v_0 \rangle = \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(v), \nabla v) D^i v dx + \int_{\Omega} |v|^{p_0} dx \geq \alpha' \sum_{i=1}^N \int_{\Omega} |D^i v|^{p_i} dx$$

with $\alpha' = \min \alpha$

It follows that



$$\begin{aligned} \frac{\langle A_n v, v - v_0 \rangle}{\|v\|_{1,\bar{p}}} &= \frac{\alpha'}{\|v\|_{1,\bar{p}}} \sum_{i=1}^N \int_{\Omega} |D^i v|^{p_i} dx - \frac{\|v_0\|_{1,\bar{p}}}{\|v\|_{1,\bar{p}}} \left(\int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \\ &+ C_{15} \frac{\|v_0\|_{1,\bar{p}}}{\|v\|_{1,\bar{p}}} \sum_{i=1}^N \left(\int_{\Omega} (R_i(x)^{p'_i} + n^{(p-1)p'_i} + |D^i v|^{p_i}) \right)^{\frac{1}{p'_i}} \|v_0\|_{1,\bar{p}} \\ &+ \frac{\|v_0\|_{1,\bar{p}}}{\|v\|_{1,\bar{p}}} \left(\int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \end{aligned} \quad (5.1)$$

Using the *Holdër's* type inequality, we have for all $u, v \in W_0^{1,\bar{p}}(\Omega)$,

$$\begin{aligned} \langle G_n v, v \rangle &= \int_{\Omega} g_n(x, v, \nabla v) v dx \leq \|g_n(x, v, \nabla v)\|_{p'_i} \|v\|_{p_i} \\ &\leq \left(\int_{\Omega} |g_n(x, v, \nabla v)|^{p'_i} + 1 \right)^{\frac{1}{p'_i}} \|v\|_{1,\bar{p}} \\ &\leq (n^{p'_i} \text{meas}(\Omega) + 1)^{\frac{1}{p'_i}} \|v\|_{1,\bar{p}} \leq C_{16} \|v\|_{1,\bar{p}} \end{aligned} \quad (5.2)$$

$$\begin{aligned} \langle G_n v, v_0 \rangle &= \int_{\Omega} g_n(x, v, \nabla v) v_0 dx \\ &\leq \left(\int_{\Omega} |g(x, v, \nabla v)|^{p'_i} \right)^{\frac{1}{p'_i}} \|v_0\|_{p_i} \\ &\leq C_{17} \sum_{i=1}^N \left(\int_{\Omega} (c(x)^{p'_i} + |D^i v|^{p_i}) \right)^{\frac{1}{p'_i}} \|v_0\|_{1,\bar{p}} \end{aligned} \quad (5.3)$$

It follows that

$$\frac{\langle G_n v, v - v_0 \rangle}{\|v\|_{1,\bar{p}}} = C_{16} \frac{\|v_0\|_{1,\bar{p}}}{\|v\|_{1,\bar{p}}} - C_{17} \frac{\|v_0\|_{1,\bar{p}}}{\|v\|_{1,\bar{p}}} \sum_{i=1}^N \left(\int_{\Omega} (c(x)^{p'_i} + |D^i v|^{p_i}) \right)^{\frac{1}{p'_i}} \quad (5.4)$$

From (5.1) and (5.4), as $\|v\|_{1,\bar{p}} \rightarrow \infty$, we conclude that

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1,\bar{p}}} = \frac{\langle A_n v, v - v_0 \rangle}{\|v\|_{1,\bar{p}}} + \frac{\langle G_n v, v - v_0 \rangle}{\|v\|_{1,\bar{p}}} \rightarrow \infty \quad (5.5)$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $W_0^{1,\bar{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W^{1,\bar{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{1,\bar{p}'}(\Omega) \\ \limsup_{k \rightarrow \infty} \langle B_n u_k, u_k \rangle \leq \langle \chi_n, u \rangle \end{cases} \quad (5.6)$$

We will prove that $\chi_n = B_n u$ and $\langle B_n u_k, u_k \rangle \rightarrow \langle \chi_n, u \rangle$ as $k \rightarrow \infty$

Firstly, in view of the compact embedding $W^{1,\bar{p}}(\Omega) \hookrightarrow L^p(\Omega)$ we have $u_k \rightarrow u$ in $L^p(\Omega)$ for a subsequence denoted again $(u_k)_{k \in \mathbb{N}}$.





As $(u_k)_{k \in \mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}}(\Omega)$, then by the growth condition $(a_i(x, T_n(u_k), \nabla u_k))_{k \in \mathbb{N}}$ is bounded in $L^{p'_i}(\Omega)$. Therefore, there exists a function $\varphi_i \in L^{p'_i}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{in} \quad L^{p'_i}(\Omega) \quad \text{as} \quad k \rightarrow \infty \quad (5.7)$$

and we have

$$|u_k|^{p_0-2} u_k \rightarrow |u|^{p_0-2} u \quad \text{in} \quad L^{p'_0}(\Omega) \quad \text{as} \quad k \rightarrow \infty \quad (5.8)$$

On the one hand, we have for all $v \in W^{1,\vec{p}}_0(\Omega)$

$$\langle B_n u_k, u_k \rangle \rightarrow \langle \chi_n, u \rangle \quad \text{as} \quad k \rightarrow \infty$$

$$\begin{aligned} \langle \chi_n, v \rangle &= \lim_{k \rightarrow \infty} \langle B_n u_k, u_k \rangle \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i v \, dx + \lim_{k \rightarrow \infty} \int_{\Omega} |u_k|^{p_0-2} u_k v \, dx \\ &= \sum_{i=1}^N \int_{\Omega} \varphi_i D^i v \, dx + \int_{\Omega} |u|^{p_0-2} u v \, dx \end{aligned} \quad (5.9)$$

From relations (5.6) and (5.9), we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle B_n u_k, u_k \rangle &= \limsup_{k \rightarrow \infty} \left(\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_{\Omega} |u_k|^{p_0} \, dx \right) \\ &\leq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u \, dx + \int_{\Omega} |u|^{p_0} \, dx. \end{aligned} \quad (5.10)$$

On the other hand, by (3.3) we get

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u)) (D^i u_k - D^i u) \, dx \\ + \lim_{k \rightarrow \infty} \int_{\Omega} (|u_k|^{p_0-2} u_k - |u|^{p_0-2} u) (u_k - u) \, dx \geq 0 \end{aligned} \quad (5.11)$$

Then

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \\ + \int_{\Omega} |u_k|^{p_0} \, dx \geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u \, dx \\ + \int_{\Omega} |u_k|^{p_0-2} u_k u \, dx + \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u) (D^i u_k - D^i u) \, dx \\ + \int_{\Omega} |u|^{p_0-2} u (u_k - u) \, dx \end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have





$T_n(u_k) \rightarrow T_n(u)$ in $L^{p_i}(\Omega)$ then $a_i(x, T_n(u_k), \nabla u) \rightarrow a_i(x, T_n(u), \nabla u)$ in $L^{p_i}(\Omega)$ and by (5.7), we get

$$\liminf_{k \rightarrow \infty} \left(\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx + \int_{\Omega} |u_k|^{p_0} dx \right) \geq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u dx + \int_{\Omega} |u|^{p_0} dx, \quad (5.12)$$

and

$$\int_{\Omega} g_n(x, u_k, \nabla u_k) u_k dx \rightarrow \int_{\Omega} \phi_i u dx \quad (5.13)$$

Having in mind (5.10) and (5.12) we conclude that

$$\lim_{k \rightarrow \infty} \left(\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx + \int_{\Omega} |u_k|^{p_0} dx \right) = \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u dx + \int_{\Omega} |u|^{p_0} dx, \quad (5.14)$$

According to (5.8), (5.13) and (5.14), we obtain

$$\langle B_n u_k, u_k \rangle \rightarrow \langle \chi_n, u \rangle \text{ as } k \rightarrow \infty$$

Now, by (5.14) we can prove that

$$\lim_{k \rightarrow \infty} \left(\sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u), \nabla u)) (D^i u_k - D^i u) dx + \int_{\Omega} (|u_k|^{p_0-2} u_k - |u|^{p_0-2} u) (u_k - u) dx \right) = 0$$

So, by virtue of **Lemma 3.2**, we get

$$u_k \rightarrow u \text{ in } W_0^{1, \vec{p}}(\Omega) \text{ and } D^i u_k \rightarrow D^i u \text{ a.e. in } \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightarrow a_i(x, T_n(u), \nabla u) \text{ in } L^{p'_i}(\Omega) \text{ for } i = 1, \dots, N, \text{ and}$$

$$g_n(x, u_k, \nabla u_k) \rightarrow g_n(x, u, \nabla u) \text{ in } L^{p'_0}(\Omega),$$

which implies that $\chi = B_n u$, which completes the proof of **lemma 5.1**.

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