التحليل والنتائج العددية لمؤثر ب . لابلاس على مسائل ستكلوف، نيومان

Analysis and Numerical Results of p-Laplacian Effect on Steklov Neumann Problems

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الملخص

شرحت هذه الورقة كيفية الحصول على وجود وتفرد الحلول الضعيفة للقيمة المرجحة لحدود ب . لابلاس لمسائل ستكلوف، نيومان ، بالاعتماد على نظرية براودر، في ظل ظروف الرتيبة للدالة f. كما ناقشت سلوك المسائل من خلال النتائج العددية. بالإضافة إلى ذلك، لتوضيح نهجنا تم تقديم بعض الأمثلة على المعادلات الخطية والغير خطية في الورقة.

الكلمات المفتاحية: مؤثر ب . لابلاس ، مسائل ستكلوف . نيومان ، النتائج العددية.

Abstract

This paper explains how to obtain the existence and uniqueness of weak solutions for the weighted p-Laplacian boundary problem of Steklov and Neumann, relying on Browder's theorem under conditions of the monotonous function f. The paper also discusses the behavior of the problems through the numerical results. Additionally, to illustrate the effectiveness of our approach, examples of both linear and nonlinear equations are presented.

Key Words: p-Laplacian operator, Steklov, Neumann problems, Numerical results.

1 Introduction

As known, p-Laplacian operator appears in pure mathematics such as applied mathematics and problems of curves. Furthermore, it intervenes in numerous fields in experimental sciences: nonlinear reaction-diffusion problems, non-Newtonian fluids flows, dynamics of populations, nonlinear elasticity and petroleum extraction [(Diaz 1985)], flows through porous medias, torsional creep problems etc.
Even though, the p-Laplacian equation has been studied widely over the past fifty years, some problems remain open. Among the previous studies from 1980 to 1989 to name but a few see [(Otani 1984), (Thelin 1984), (Serrin 1986), (Otani 1988), (Thelin 1987), (Sakaguchi 1987), (Anane 1987) and (Barles 1988)], from 1990 to 2000 see [(Lindqvist 1990), (Anane and Gossez 1990), (Manasevich and Del Pino 1991) , (Drabek 1995),(Drabek and Huang 1997) and (Bechah, Chaib and Thelin 2000)].
Moreover, the study of partial differential equations and variational problems in the presence of variable exponents has been a successful in recent years, see for example [(Mihailescu 2007), (Dend 2008), (El Amrouss, Moradi and Moussaoui 2009) and (Ayoujil and El Amrouss 2011)].
On the other hand, the numerical solution of singularly perturbed boundary value problems has recently received much attention and still constitutes a very active research area, as evidenced by research articles and books, describing various methods for solving singular perturbation problems see for example [(Kumar, Srivastava and Kumar Singh 2012), (Hssini, Massar, Talbi and Tsouli 2014), (Lindqvist 2005), (Neuberger and Swift 2001), (Torné 2005), (Weng, Zhai and Feng 2015) and (El-zahrani and Serag)].

In this paper, we are interested in proving the existence and uniqueness of weak solutions of the two strongly nonlinear elliptic problems where the diffusion term is the p Laplacian operator which can be written as follows:

$$(P_{i,1})\begin{cases} -\Delta_p u + a_i(x)|u|^{p-2}u = f_i(x,u) & \text{in } \Omega \\ (\gamma|\nabla u|^{p-2} + (1-\gamma))\frac{\partial u}{\partial \nu} = g_1(x,u) & \text{on } \partial\Omega, \end{cases}$$

$i = 1$ and 2, $\gamma = 0$ and 1. Let Ω be a bounded domain in $\mathbb{R}^N$, $(N \geq 2)$, $(1 < p < +\infty)$.
$(P_{1,1})$ is the first problem (P_1) such that $a_1(x) \equiv 1$, $f_1(x,u) \equiv 0$ and $g_1(x,u) = f(x,u)$.
$(P_{2,0})$ is the second problem (P_2) such that $0 < \alpha \leq a_2(x) \equiv a(x) \leq \beta < +\infty$, $f_2(x,u) = f(x,u)$ and $g_2(x,u) \equiv 0$.
The non linearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory (CAR) Satisfies the following :
(H_1) $f(x,s_1) \leq f(x,s_2)$ for a.e $x \in \partial\Omega$ and $\forall s_1, s_2 \in \mathbb{R}$, $s_1 \geq s_2$.
(H_2) $|f(x,s)| \leq f_0(x) + c|s|^{p-1}$ for a.e $x \in \partial\Omega$ for all $s \in \mathbb{R}$
that there exists $f_0 \in L^{p'}(\partial\Omega)$, $c > 0$.

The rest of the paper is organized as follows: section 2 addresses the main result on the existence and uniqueness of weak solutions of the problems (P_1) and (P_2), some examples as applications of our results are presented in Section 3.Finally, in section 4, we discussed the numerical results with Steklov or Neumann boundary conditions using MATLAB.

2 Existence and uniqueness results

In this section, we are going to prove the existence and uniqueness of weak solution for equations (P_1) and (P_2) using Browder theorem.

Definition 2.1 *We say that $u \in W^{1,p}(\Omega)$ is a weak solution to equation (P_1) if*
$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega |u|^{p-2}u\varphi dx = \int_{\partial\Omega} f(x,u)\varphi dx \ \forall \varphi \in W^{1,p}(\Omega)$.

Definition 2.2 *We say that $u \in W^{1,p}(\Omega)$ is a weak solution to equation (P_2) if* $\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega a(x)|u|^{p-2}u\varphi dx = \int_\Omega f(x,u)\varphi dx,\ \forall \varphi \in W^{1,p}(\Omega)$.

Our main results concerning problems (P_1) and (P_2) are the following theorems :

Theorem 2.1 *Let $p \geq 2$, and $f(x,u) \in CAR(\Omega \times \mathbb{R})$ satisfy (H_1) and (H_2). Then problem (P_1) has a unique weak solution.*

Proof :
We define for the operator $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$, as $A = J - F$, where the operators $J : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$, and $F : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ are given by

$$\langle J(u), \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega |u|^{p-2}u\varphi dx$$

and

$$\langle F(u), \varphi\rangle = \int_{\partial\Omega} f(x,u)\varphi dx$$

for all $u, \varphi \in W^{1,p}(\Omega)$. Thus, to find a weak solution of (P_1) is equivalent to finding $W^{1,p}(\Omega)$ which satisfies the operator equation $A(u) = 0$. Following, the properties of the operators J and F:
a) J and F are well defined. Using Hölders inequality, we have $\langle J(u), \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_\Omega |u|^{p-2}u\varphi dx$
$|\langle J(u), \varphi\rangle| \leq \int_\Omega |\nabla u|^{p-1}|\nabla\varphi| dx + \int_\Omega |u|^{p-1}|\varphi| \leq \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla\varphi|^p\right)^{\frac{1}{p}} + \left(\int_\Omega |u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} < \infty.$
And
$\langle F(u), \varphi\rangle = \int_{\partial\Omega} f(x,u)\varphi dx |\langle F(u), \varphi\rangle| \leq \int_{\partial\Omega} |f(x,u)\varphi| dx \leq \int_{\partial\Omega} (f_0(x) + c|u|^{p-1})|\varphi| dx$ By Hölders inequality, $|\langle F(u), \varphi\rangle| \leq \left(\int_{\partial\Omega} |f_0(x)|^{p'}\right)^{\frac{1}{p'}} \left(\int_{\partial\Omega} |\varphi|^p\right)^{\frac{1}{p}} + c\left(\int_{\partial\Omega} |u|^p\right)^{\frac{1}{p'}} \left(\int_{\partial\Omega} |\varphi|^p\right)^{\frac{1}{p}} < \infty$ And hence J et F are well defined.

b) J and F are bounded operators. Indeed, for every u such that
$\|u\|_{W^{1,p}(\Omega)} \leq M$ We have
$\|J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \leq 1} |\langle J(u), \varphi\rangle| \leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \leq 1} \left(\int_\Omega |\nabla u|^{p-1}|\nabla\varphi| dx + \int_\Omega |u|^{p-1}|\varphi|\right) dx.$

$$\|J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \leq 1} \left(\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla\varphi|^p\right)^{\frac{1}{p}} + \left(\int_\Omega |u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}}\right)$$

$$= \sup_{\|\varphi\| \leq 1} \left(\|\nabla u\|_{L^p}^{\frac{p}{p'}} \|\varphi\|_{L^p} + \|u\|_{L^p}^{\frac{p}{p'}} \|\varphi\|_{L^p}\right)$$

$\leq \|\nabla u\|_{L^p}^{\frac{p}{p'}} + \|u\|_{L^p}^{\frac{p}{p'}} \leq \|u\|^{\frac{p}{p'}} + \|u\|^{\frac{p}{p'}} 2\|u\|^{\frac{p}{p'}} \leq 2M^{\frac{p}{p'}}$. Also, we get

$$\|F(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\partial\Omega)} \leq 1} |\langle F(u), \varphi\rangle| \leq \sup_{\|\varphi\|_{W^{1,p}(\partial\Omega)} \leq 1} \int_{\partial\Omega} (f_0(x) + c|u|^{p-1})|\varphi|$$

$$\leq \sup_{\|\varphi\|_{W^{1,p}(\partial\Omega)} \leq 1}$$

$$\left[\left(\int_{\partial\Omega} |f_0(x)|^{p'}\right)^{\frac{1}{p'}} + \left(\int_{\partial\Omega} |u|^{(p-1)p'}\right)^{\frac{1}{p'}}\right] \left(\int_{\partial\Omega} |\varphi|^p\right)^{\frac{1}{p}}.$$

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$$\leq k(\|f_0\|_{L^{p'}(\Omega)} + k\|u\|^{\frac{p}{p'}}_{W^{1,p}(\Omega)}) \leq k(\|f_0\|_{L^{p'}(\Omega)} + kM^{\frac{p}{p'}})$$

Where k is the constant of the embedding of $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$,.

c) J and F are continuous operators. If $u_n \to u$, in $W^{1,p}(\Omega)$. Then, we have $\|u_n - u\|_{W^{1,p}(\Omega)} \to 0$, $\|\nabla u_n - \nabla u\|_{L^p(\Omega)} \to 0, \|u_n - u\|_{L^p} \to 0$ Applying Dominated Convergence Theorem, we obtain $\|(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\|_{L^p(\Omega)} \to 0$ and $\|(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^p(\Omega)} \to 0$ Hence
$\|J(u_n) - J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} |\langle J(u_n) - J(u), \varphi\rangle|$

$$\leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} \left(\left(\int_\Omega \left([|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u] \right)^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} \right.$$

$$\left. + \left(\int_\Omega \left(([|u_n|^{p-2}u_n - |u|^{p-2}u]) \right)^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} \right) \to 0 \text{ as } n \to \infty.$$

Similarly, we have
$\|F(u_n) - F(u)\|_{(W^{1,p}(\partial\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\partial\Omega)}\leq 1} |\langle F(u_n) - F(u), \varphi\rangle| \leq k \left(\int_{\partial\Omega} |f(x,u_n) - f(x,u)|^{p'} \right)^{\frac{1}{p'}} \to 0$ as $n \to \infty$.

d) Let $p \geq 2$, $\forall x_1, x_2 \in \mathbb{R}^{\mathbb{N}}$, we have the following inequality

$$|x_2|^p \geq |x_1|^p + p|x_1|^{p-2}x_1(x_2 - x_1) + \frac{|x_2 - x_1|^p}{2^{p-1} - 1}. \tag{1}$$

Now,

$$\langle J(u) - J(\varphi), u - \varphi\rangle = \int_\Omega \left[|\nabla u|^{p-2}\nabla u - |\nabla\varphi|^{p-2}\nabla\varphi \right] (\nabla u - \nabla\varphi)$$

$$+ \int_\Omega \left[|u|^{p-2}u - |\varphi|^{p-2}\varphi \right] (u - \varphi)$$

$$= \int_\Omega [|\nabla u|^{p-2}\nabla u(\nabla u - \nabla\varphi)] - \int_\Omega [|\nabla\varphi|^{p-2}\nabla\varphi(\nabla u - \nabla\varphi)]$$

$$+ \int_\Omega [|u|^{p-2}u(u - \varphi)] - \int_\Omega [|\varphi|^{p-2}\varphi(u - \varphi)] = I_1 + I_2.$$

Using (1), we get

$$I_1 + I_2 \geq \frac{2}{p(2^{p-1} - 1)} \int_\Omega |\nabla u - \nabla\varphi|^p dx + \frac{2}{p(2^{p-1} - 1)} \int_\Omega |u - \varphi|^p dx$$

$$\geq c(p) \left(\|\nabla u - \nabla\varphi\|^p_{L^p(\Omega)} + \|u - \varphi\|^p_{L^p(\Omega)} \right)$$

$$= c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)}, \text{ for } p \geq 2.$$

So

$$\langle J(u) - J(\varphi), u - \varphi\rangle \geq c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)} \text{ for } p \geq 2. \tag{2}$$

Also, we get

$$\langle F(u) - F(\varphi), u - \varphi\rangle = \int_{\partial\Omega} [f(x,u) - f(x,\varphi)](u - \varphi).$$

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Since f is decreasing with respect to the second variable, we have

$$[f(x,u)-f(x,\varphi)](u-\varphi)\leq 0.$$

Consequently

$$\langle F(u)-F(\varphi),u-\varphi\rangle=\int_{\partial\Omega}[f(x,u)-f(x,\varphi)](u-\varphi)\leq 0. \tag{3}$$

Equations (2) and (3) imply that

$$\langle A(u)-A(\varphi),u-\varphi\rangle\geq c(p)\|u-\varphi\|^p_{W^{1,p}(\Omega)} \text{ for } p\geq 2. \tag{4}$$

So A is strongly monotone. Now, to apply Browder theorem, it remains to prove that A is a coercive operator. From (4) , we have

$$\langle A(u),u\rangle\geq\langle A(0),u\rangle+c(p)\|u\|^p_{W^{1,p}(\Omega)}.$$

On the other hand

$$\begin{aligned}\langle A(0),u\rangle&=\langle J(0),u\rangle-\langle F(0),u\rangle\\&=-\int_{\partial\Omega}f(x,0)u\\&\geq-(\int_{\partial\Omega}[f_0(x)]^{p'})^{\frac{1}{p'}}(\int_{\partial\Omega}|u|^p)^{\frac{1}{p}}\\&\geq-k\|f_0\|_{L^{p'}(\partial\Omega)}\|u\|_{W^{1,p}(\partial\Omega)}.\end{aligned}$$

Then

$$\langle A(u),u\rangle\geq c(p)\|u\|^p_{W^{1,p}(\Omega)}-k\|f_0\|_{L^{p'}(\partial\Omega)}\|u\|_{W^{1,p}(\partial\Omega)}.$$

So

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty}\frac{\langle A(u),u\rangle}{\|u\|_{W^{1,p}(\Omega)}}=\infty.$$

So,after proving the coercivity condition. Thus the existence of weak solution for (P_1). The uniqueness of weak solution of (P_1) is a direct consequence of (4). Suppose that u,φ be a weak solutions of (P_1) such that $u\neq\varphi$. Now, from (4), we have

$$0=\langle A(u)-A(\varphi),u-\varphi\rangle\geq c(p)\|u-\varphi\|^p_{W^{1,p}(\Omega)}\geq 0.$$

Therefore $u=\varphi$ This completes the proof.

Theorem 2.2 *Let $p\geq 2$,and $f(x,u)\in CAR(\Omega,\mathbb{R})$ satisfy (H_1) and (H_2). Then problem (P_2) has a unique weak solution.*

Proof:
We define for the operator $A:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))'$, as $A=J-F$, where the operators $J:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))'$, and $F:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))'$ are given by

$$\langle J(u),\varphi\rangle=\int_\Omega|\nabla u|^{p-2}\nabla u\nabla\varphi dx+\int_\Omega a(x)|u|^{p-2}u\varphi dx$$

and

$$\langle F(u),\varphi\rangle=\int_\Omega f(x,u)\varphi dx$$

for all $u, \varphi \in W^{1,p}(\Omega)$. Thus, to find a weak solution of (P) is equivalent to finding $W^{1,p}(\Omega)$ which satisfies the operator equation $A(u) = 0$. Following, the properties of the operators J and F
a) J and F are well defined. Using Hölders inequality, we have

$$\langle J(u), \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \int_\Omega a(x)|u|^{p-2} u\varphi dx$$

$$|\langle J(u), \varphi \rangle| \le \int_\Omega |\nabla u|^{p-1} |\nabla \varphi| dx + \int_\Omega a(x)|u|^{p-1}|\varphi|$$

$$\le \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla \varphi|^p\right)^{\frac{1}{p}} + \beta \left(\int_\Omega |u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} < \infty.$$

And

$$\langle F(u), \varphi \rangle = \int_\Omega f(x,u)\varphi dx$$

$$|\langle F(u), \varphi \rangle| \le \int_\Omega |f(x,u)\varphi| dx \le \int_\Omega (f_0(x) + c|u|^{p-1})|\varphi| dx$$

By Hölders inequality,

$$|\langle F(u), \varphi \rangle| \le \left(\int_\Omega |f_0(x)|^{p'}\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} + c\left(\int_\Omega |u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} < \infty$$

And hence J, F are well defined.
b) J and F are bounded operators. Indeed, for every u such that $\|u\|_{W^{1,p}(\Omega)} \le M$

$\|J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \le 1} |\langle J(u), \varphi \rangle| \le \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \le 1} \left(\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla \varphi|^p\right)^{\frac{1}{p}} + \beta \left(\int_\Omega |u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} \right)$
$= \sup_{\|\varphi\| \le 1} \left(\|\nabla u\|_{L^p}^{\frac{p}{p'}} \|\varphi\|_{L^p} + \beta \|u\|_{L^p}^{\frac{p}{p'}} \|\varphi\|_{L^p} \right) \le \|\nabla u\|_{L^p}^{\frac{p}{p'}} + \beta \|u\|_{L^p}^{\frac{p}{p'}} \le \|u\|^{\frac{p}{p'}} + \beta \|u\|^{\frac{p}{p'}} 2\beta \|u\|^{\frac{p}{p'}} \le 2\beta M^{\frac{p}{p'}}$. Also, we get
$\|F(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \le 1} |\langle F(u), \varphi \rangle| \le \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \le 1} \int_\Omega (f_0(x) + c|u|^{p-1})|\varphi|$

$$\le \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \le 1} [\left(\int_\Omega |f_0(x)|^{p'}\right)^{\frac{1}{p'}} + \left(\int_\Omega |u|^{(p-1)p'}\right)^{\frac{1}{p'}}] \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}}.$$

$$\le k(\|f_0\|_{L^{p'}(\Omega)} + k\|u\|_{W^{1,p}(\Omega)}^{\frac{p}{p'}})$$

$$\le k(\|f_0\|_{L^{p'}(\Omega)} + kM^{\frac{p}{p'}})$$

Where k is the constant of the embedding of $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$.
c) J and F are continuous operators if $u_n \to u$, in $W^{1,p}(\Omega)$. Then, we have

$$\|u_n - u\|_{W^{1,p}(\Omega)} \to 0, \ \|\nabla u_n - \nabla u\|_{L^p(\Omega)} \to 0, \|u_n - u\|_{L^p} \to 0$$

Applying Dominated Convergence Theorem, we obtain

$$\|(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u)\|_{L^p(\Omega)} \to 0$$

and

$$\|(|u_n|^{p-2} u_n - |u|^{p-2} u)\|_{L^p(\Omega)} \to 0$$

Hence

$$\|J(u_n) - J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} |\langle J(u_n) - J(u), \varphi\rangle|$$

$$\leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} \left(\left(\int_\Omega \left([|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u]\right)^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} \right.$$

$$\left. +\beta \left(\int_\Omega (([|u_n|^{p-2}u_n - |u|^{p-2}u)])^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} \right) \to 0 \text{ as } n \to \infty.$$

Similarly, we have

$$\|F(u_n) - F(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} |\langle F(u_n) - F(u), \varphi\rangle|$$

$$\leq k \left(\int_\Omega |f(x,u_n) - f(x,u)|^{p'} \right)^{\frac{1}{p'}} \to 0 \text{ as } n \to \infty.$$

d) Let $p \geq 2$, $\forall x_1, x_2 \in \mathbb{R}^N$, we have the following inequality

$$|x_2|^p \geq |x_1|^p + p|x_1|^{p-2}x_1(x_2 - x_1) + \frac{|x_2 - x_1|^p}{2^{p-1} - 1}. \tag{5}$$

Now, $\langle J(u) - J(\varphi), u - \varphi\rangle = \int_\Omega \left[|\nabla u|^{p-2}\nabla u - |\nabla\varphi|^{p-2}\nabla\varphi\right](\nabla u - \nabla\varphi) + \int_\Omega a(x) \left[|u|^{p-2}u - |\varphi|^{p-2}\varphi\right](u - \varphi)$

$$= \int_\Omega [|\nabla u|^{p-2}\nabla u(\nabla u - \nabla\varphi)] - \int_\Omega [|\nabla\varphi|^{p-2}\nabla\varphi(\nabla u - \nabla\varphi)]$$

$$+ \int_\Omega a(x)[|u|^{p-2}u(u - \varphi)] - \int_\Omega a(x)[|\varphi|^{p-2}\varphi(u - \varphi)] = I_1 + I_2.$$

Using (5), we get $I_1 + I_2 \geq \frac{2}{p(2^{p-1}-1)} \int_\Omega |\nabla u - \nabla\varphi|^p dx + \frac{2}{p(2^{p-1}-1)} \int_\Omega a(x)|u - \varphi|^p dx$

$$\geq \alpha c(p) \left(\|\nabla u - \nabla\varphi\|^p_{L^p(\Omega)} + \|u - \varphi\|^p_{L^p(\Omega)} \right)$$

$$= \alpha c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)}, \text{ for } p \geq 2.$$

So

$$\langle J(u) - J(\varphi), u - \varphi\rangle \geq \alpha c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)} \text{ for } p \geq 2. \tag{6}$$

Also, we get

$$\langle F(u) - F(\varphi), u - \varphi\rangle = \int_\Omega [f(x,u) - f(x,\varphi)](u - \varphi).$$

Since f is decreasing with respect to the second variable, we have

$$[f(x,u) - f(x,\varphi)](u - \varphi) \leq 0.$$

Consequently

$$\langle F(u) - F(\varphi), u - \varphi\rangle = \int_\Omega [f(x,u) - f(x,\varphi)](u - \varphi) \leq 0. \tag{7}$$

Equations (6) and (7) imply that

$$\langle A(u) - A(\varphi), u - \varphi\rangle \geq c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)} \text{ for } p \geq 2. \tag{8}$$

So A is strongly monotone. Now, to apply Browder theorem, it remains to prove that A is a coercive operator. From (8) , we have

$$\langle A(u), u\rangle \geq \langle A(0), u\rangle + c(p)\|u\|^p_{W^{1,p}(\Omega)}.$$

On the other hand

$$\begin{aligned}\langle A(0), u\rangle &= \langle J(0), u\rangle - \langle F(0), u\rangle \\ &= -\int_\Omega f(x,0)u \\ &\geq -(\int_\Omega [f_0(x)]^{p'})^{\frac{1}{p'}}(\int_\Omega |u|^p)^{\frac{1}{p}} \\ &\geq -k\|f_0\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)}.\end{aligned}$$

Then

$$\langle A(u), u\rangle \geq c(p)\|u\|^p_{W^{1,p}(\Omega)} - k\|f_0\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)}.$$

So

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty} \frac{\langle A(u), u\rangle}{\|u\|_{W^{1,p}(\Omega)}} = \infty.$$

This proves the coercivity condition and so, the existence of weak solution for (P_2). The uniqueness of weak solution of (P_2) is a direct consequence of (8). Suppose that u, φ be a weak solutions of (P) such that $u \neq \varphi$. Now, from (8), we have

$$0 = \langle A(u) - A(\varphi), u - \varphi\rangle \geq c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)} \geq 0.$$

Therefore $u = \varphi$ This completes the proof.

3 Examples

To illustrate the effectiveness of our approach, we provide the following examples of linear and non-linear to problem (P_1).

Exemple 3.1 *Let $a, b \in \mathbb{R}$, $a < b$ let $u : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$. We pose*

$$f(x, u(x)) = -u(x) + 1,$$

than the function $f(x, u)$ is decreasing with respect to the second variable,

$$u'(a) = f(a, u(a)) = -u(a) + 1$$

$$u'(b) = f(b, u(b)) = -u(b) + 1.$$

such that $u(x) = \alpha e^{-x}$ with $\alpha \in \mathbb{R}$, is a solution to the problem

$$\begin{cases} \Delta u = u \ in \ ]a, b[, \\ \frac{\partial u}{\partial \eta} = f(x, u) \ on \ \{a, b\} \end{cases}$$

because

$$\begin{cases} \Delta u(x) = -(-\alpha e^{-x}) = u(x) \\ \frac{\partial u}{\partial x}(a) = -u(a) = -\alpha e^{-a} = f(a, u(a)) \\ \frac{\partial u}{\partial x}(b) = -u(b) = -\alpha e^{-b} = f(b, u(b)) \end{cases}$$

Exemple 3.2 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that there exists $a, b, \alpha, \beta \in \mathbb{R}$ such that $a < b$ and $u(x_1, x_2) = \alpha e^{x_1} + \beta e^{x_2}$ and*

$$f((x_1,x_2),y) = \begin{cases} -\alpha e^a & \text{if } x_1 = a; \forall x_2, y \in \mathbb{R} \\ \alpha e^b & \text{if } x_1 = b; \forall x_2, y \in \mathbb{R} \\ -\beta e^a & \text{if } x_2 = a; \forall x_1, y \in \mathbb{R} \\ \beta e^b & \text{if } x_2 = b; \forall x_1, y \in \mathbb{R} \\ 0 & ifnon \end{cases}$$

Then u is a solution to the problem

$$\begin{cases} \Delta u = u & in\ ]a,b[\times]a,b[\\ \frac{\partial u}{\partial \eta} = f(x,u) & on\ \{a\} \times [a,b] \cup \{b\} \times [a,b] \end{cases}$$

Exemple 3.3 *Let $a, b \in \mathbb{R}$, $a < b$ let $u : [a,b] \to \mathbb{R}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ such that $u(x) = e^{\frac{1}{\sqrt[4]{3}}x}$*

$$f(x,y) = \begin{cases} \frac{1}{\sqrt[4]{27}}(\frac{2x-a-b}{b-a})y^3 & if\ x \in [a, \frac{a+b}{2}] \cup \{b\} \\ \frac{1}{\sqrt[4]{27}}(\frac{2x-a-b}{a-b})y^3 & if\ x \in [\frac{a+b}{2}, b] \end{cases}$$

Then u and f satisfying the problem

$$\begin{cases} \Delta_4 u = 3(u'')(u')^2(x) = u^3(x) & in\ ]a,b[\\ (u')^3(a) = -f(a, u(a)) \\ (u')^3(b) = f(b, u(b)) \end{cases}$$

4 Discretisation

To illustrate our numerical, we consider a Steklov problem (P_1). In particular, let $p = 2, \Omega =]a,b[$, we pose $f(x,u(x)) = -u(x) + 1$, than the function $f(x,u)$ is decreasing with respect to the second variable, The problem (P_1) becomes

$$\begin{cases} \Delta u = u \text{ dans }]a,b[, \\ \frac{\partial u}{\partial \eta} = f(x,u) = -u + 1 \text{ sur } \{a,b\} \end{cases} \tag{9}$$

we set $h = \frac{b-a}{n}$, $x_i = a + ih$, using Taylor expansion of $x_i = a + ih$ a second-order with $(1 \leq i \leq n-1)$ we have

$$\begin{cases} u'' = \frac{u_{i+1}+u_{i-1}-2u_i}{h^2} = u_i & 1 \leq i \leq n-1 \\ u'(a) = \frac{u(a+h)-u(a)}{h} = -u(a) + 1 = -u_0 + 1 \\ u'(b) = \frac{u(b)-u(b-h)}{h} = -u(b) + 1 = -u_n + 1 \end{cases}$$

with Matlab, we obtain see (figure (1) and (2)).
With of Neumann problem (P_2). In particular, let $p = 2, \Omega =]a,b[$, we pose $f(x,u)$ is decreasing with respect to the second variable, we pose $a(x) = x^2 + 1$, $f(x,u(x)) = -u(x)e^{-x} + 1$, The problem (P_1) becomes

$$(*) \begin{cases} \Delta u + (x^2+1)u = -ue^{-x} + 1 \text{ in }]a,b[, \\ \frac{\partial u}{\partial \eta} = 0 \text{ on } \{a,b\} \end{cases}$$

we set $h = \frac{b-a}{n}$, $x = ih$, using Taylor expansion of $x_i = a + ih$ a second-order with $(1 \leq i \leq n-1)$.
We have

$$\begin{cases} \frac{u_{i+1}+u_{i-1}-2u_i}{h^2} + ((ih)^2 + 1 + e^{-hi})u_i = 1 & 1 \leq i \leq n-1 \\ u_{i+1} + u_{i-1} + (h^2((ih)^2 + 1 + e^{-ih}) - 2)u_i = h^2 \\ u'(a) = 0 \Rightarrow \frac{u(a+h)-u(a)}{h} = 0, u_0 = u_1 \\ u'(b) = 0 \Rightarrow \frac{u(b)-u(b-h)}{h} = 0, u_n = u_{n-1} \end{cases}$$

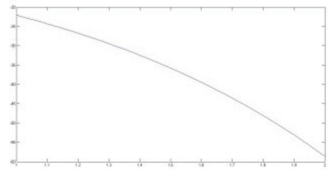
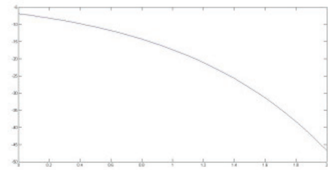


Figure 1: Steklov problem in one dimension, we take a=1, b=2 and n=50



M

Figure 2: Steklov problem in one dimension, we take a=0, b=2 and n= 50

with Matlab, we obtain see (figure (3) and (4)).

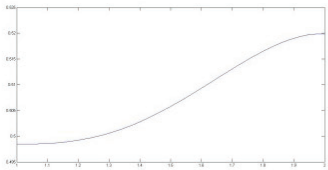


Figure 3: Neumann problem in one dimension, we take a=1, b=2 and n=50

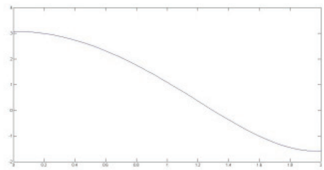


Figure 4: Neumann problem in one dimension, we take a=0, b=2 and n= 50

In the case :
$-(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) = f$ in Ω, where $\Omega : 0 \leq x \leq 1, 0 \leq y \leq 1$. We suppose that $f(x,y) = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$, we have $u(x,y) = \sin(2\pi x)\sin(2\pi y)$ is an exact solution,

$$-\Delta u = f(x,y) = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$$

we obtain the explicit scheme
$(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})|_{i,j} \approx \frac{u_{i+1,j}+u_{i-1,j}-2u_{i,j}}{\Delta x^2} + \frac{u_{i,j+1}+u_{i,j-1}-2u_{i,j}}{\Delta y^2} = f_{i,j} \Rightarrow \frac{1}{\Delta x^2}u_{i+1,j} + \frac{1}{\Delta x^2}u_{i-1,j} + \frac{1}{\Delta y^2}u_{i,j+1} + \frac{1}{\Delta y^2}u_{i,j-1} - 2(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2})u_{i,j} = f_{i,j}$, we have with the Matlab see (figure (5) and (6)).

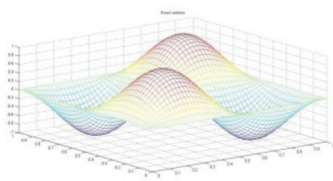


Figure 5: Neumann problem in two dimensions, exact solution

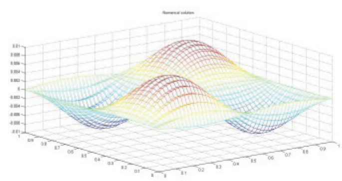


Figure 6: Neumann problem in two dimensions, numerical solution

The case nonlinear in example (2.3) we obtain the scheme

$$\begin{cases} u_{i-1} = \frac{u_i^3(dx)^4}{3(u_{i+1}-u_i)^2} - (u_{i+1} - 2u_i),\ 2 \leq i \leq n-1 \\ u_1 = u_2 + \frac{1}{\sqrt[4]{3}} e^{\frac{1}{\sqrt[4]{3}}x} dx \\ u_n = u_{n-1} + \frac{1}{\sqrt[4]{3}} e^{\frac{1}{\sqrt[4]{3}}x} dx, \end{cases}$$

we have with the Matlab see (figure (7) and (8)).

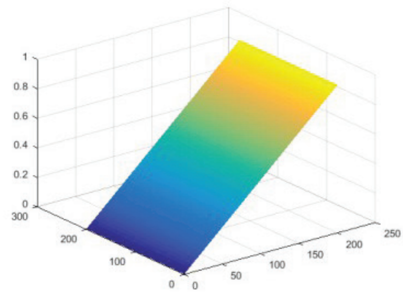


Figure 7: Steklov problem in two dimensions, exact solution

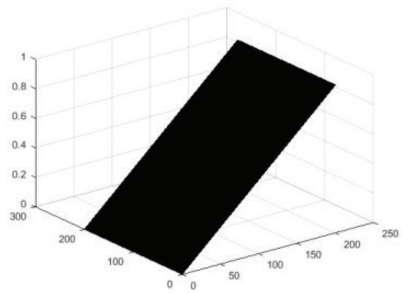


Figure 8: Steklov problem in two dimensions, numerical solution

5 Conclusion

This study explains the existence and uniqueness of weak solutions to the weighted p-Laplacian boundary problems of Steklov and Neumann. The paper also presents numerical results. Additionally, to illustrate the effectiveness of our approach, we provide three examples of both linear and nonlinear equations.

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