الحلول الكونية لمشكلة القطع الناقص الغير خطية القوية في فضاءات ميزلاك - أورلكس - سبوليف

Entropy Solutions for strong nonlinear Elliptic problem in Musielak-Orlicz-Sobolev spaces

د. محمد علي محسن الحومي[1]
Dr. Mohammed Ali Mohsen Al-Hawmi[2]
د. بسام عبد القادر عاتق الحمزة[3]
Dr. Bassam Abdulqader Ateq AL-Hamzah[4]

https://doi.org/10.54582/TSJ.2.2.18

(1) أستاذ المعادلات التفاضلية الجزئية المساعد – قسم الرياضيات –كلية التربية والعلوم– جامعة إقليم سبأ – مأرب – اليمن

عنوان المراسلة : m.alhomi2011@gmail.com:

(2) **Assistant Professor of partial differential equations, Department of Mathematics, Faculty of Education and Sciences, University of Saba Region, Marib, Yemen**

(3) أستاذ المعادلات التفاضلية الجزئية المساعد – قسم الرياضيات –كلية التربية والعلوم – جامعة إقليم سبأ – مأرب – اليمن

عنوان المراسلة : bassamalhamzah@yahoo.com

(4) **Assistant Professor of partial differential equations, Department of Mathematics, Faculty of Education and Sciences, University of Saba Region, Marib, Yemen**

الملخص

في هذه الورقة ، نبرهن وجود الحول الكونية لبعض فئة مشكلة القطع الناقص القوية الغير خطية للنوع:

$$Au + g(x, u, \nabla u) = f \quad in \;\; \Omega$$

في فضاءات ميزلاك – أورلكس – سبوليف $W_0^1L_\varphi(\Omega)$ تحت شروط أن دالة المرافق

$\psi(x,t)$ لدالة ميزلاك – أورلكس $\varphi(x,t)$ تحقق شرط Δ_2

نفرض أن الدالة $f \in L^1(\Omega)$ و $g(x,s,\xi)$ **تحقق فقط بعض النمو الغير القياسي مع ما يتعلق** $|\xi|$

الكلمات المفتاحية : فضاءات ميزلاك – أورلكس – سبوليف ، مشكلة القطع الناقص الغير خطية

،دالة القطع ، الحلول الكونية.

Abstract

In this paper, we prove the existence of entropy Solutions for some class of strong nonlinear elliptic problem of the type

$$-diva(x,u,\nabla u) + g(x,u,\nabla u) = f$$

In the Musielak–Orlicz–Sobolev spaces $W_0^1 L_\varphi(\Omega)$ under the assumption that $\psi(x,t)$, the conjugate function of the Munielak–Orlicz function $\varphi(x,t)$, satisfies the Δ_2 - condition,
we assume that $f \in L^1(\Omega)$ and $g(x,s,\xi)$ satisfies only some nonstandard growth with respect to $|\xi|$.
keywords: **Musielak–Orlicz–Sobolev spaces, nonlinear elliptic problem, truncation function, entropy solutions.**

1 Introduction

Let Ω be an open subset of $I\!R^n$. In this paper, we study the existence of solutions for strong nonlinear elliptic problems of the form:

$$Au + g(x, u, \nabla u) = f \qquad \text{in} \quad \Omega \tag{1.1}$$

where A is the Leray-Lions operator defined as:

$$A(u) = -\text{div } a(x, u, \nabla u)$$

and $g(x, s, \xi)$ presents the nonlinearity of the problem (1.1).

A. Bensoussan, L. Boccardo and F. Murat [14] proved the existence of solutions for the Dirichlet problem of the form (1.1), where $g(x, s, \xi)$ satisfies :

$$\begin{array}{rcll} |g(x,s,\xi)| & \leq & b(|s|)(c(x) + |\xi|^p) & \text{(natural growth condition)} \\ g(x,s,\xi).s & \geq & 0 & \text{(sign condition)} \end{array}$$

we refer also to [4, 5] for more details. A. Benkirane and A. Elmahi [10] have proved the existence theorem of the problem (1.1) in Orlicz-Sobolev-space $W^1L_M(\Omega)$, by assuming a sign condition and a natural growth condition on $g(x, s, \xi)$ of the form :

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M\left(\frac{|\xi|}{\lambda}\right)) \qquad (M(\cdot) \text{ is an } N\text{-function})$$

The N-function $M(\cdot)$ is supposed to satisfy the Δ_2-condition and the domain Ω of $I\!R^n$ satisfying the segment property, in order to construct a complementary system $(W_0^1L_M(\Omega), W_0^1E_M(\Omega); W^{-1}L_{\overline{M}}(\Omega), W^{-1}E_{\overline{M}}(\Omega))$.

In [15], A. Elmahi and D. Meskine have proved the existence of solution for the problem (1.1) without assuming Δ_2-condition on φ and its conjugate fonction. In [6, 8] were studies in in Musielak-Orlicz-Sobolev Spaces

The purpose of this paper is to prove the existence of entropy solutions for the strong nonlinear elliptic problem, by assuming that the conjugate function of Musielak-Orlicz function $\varphi(x, t)$ satisfies Δ_2-condition and using corollary 1 of [12] to construct a complementary system $(W_0^1L_\varphi(\Omega), W_0^1E_\varphi(\Omega); W^{-1}L_\psi(\Omega), W^{-1}E_\psi(\Omega))$ without consider that Ω verifies the segment property.

2 Preliminaries

In this section, we introduce some definitions and known facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [19].

2.1 Musielak-Orlicz function

Let Ω be an open subset of $I\!R^N$ $(N \geq 2)$, and let $\varphi(x, t)$ be a real-valued function defined in $\Omega \times I\!R^+$ and satisfying the following conditions:

(a) $\varphi(x, \cdot)$ is an N-function, *i.e.* convex, nondecreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$, and :

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0 \quad , \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$$

(b) $\varphi(\cdot, t)$ is a measurable function.

A function $\varphi(x, t)$ which satisfies conditions (a) and (b) is called a Musielak-Orlicz function.
For a Musielak-Orlicz function $\varphi(x, t)$ we set $\varphi_x(t) = \varphi(x, t)$ and let $\varphi_x^{-1}(t)$ the reciprocal function with respect to t of $\varphi_x(t)$, i.e.

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions $\varphi(x, t)$ and $\gamma(x, t)$, we introduce the following ordering:

(c) If there exists two positives constants c and T such that for almost everywhere $x \in \Omega$:

$$\varphi(x,t) \le \gamma(x,ct) \quad \text{for} \quad t \ge T,$$

we write $\varphi \prec \gamma$, and we say that γ dominate φ globally if $T = 0$, and near infinity if $T > 0$.

(d) For every positive constant c and almost everywhere $x \in \Omega$, if

$$\lim_{t\to 0}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \quad or \quad \lim_{t\to\infty}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0,$$

we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.

Let

$$\psi(x,s) = \sup_{t\ge 0}\{st - \varphi(x,t)\}$$

that is, $\psi(x,t)$ is the Musielak-Orlicz function complementary to (or conjugate) of $\varphi(x,t)$ in the sense of Young with respect to the variable s.

The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the Δ_2-condition if, there exists $k > 0$ and a nonnegative function $h(\cdot) \in L^1(\Omega)$, such that

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \qquad \text{a.e.} \quad x \in \Omega,$$

for large values of t, or for all values of t.

2.2 Musielak-Orlicz Lebesgue space

In the following, the measurability of a function $u : \Omega \longmapsto I\!R$ means the Lebesgue measurability. We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx$$

where $u : \Omega \longmapsto I\!R$ is a measurable function. The set

$$K_\varphi(\Omega) = \{u : \Omega \longmapsto I\!R \quad \text{measurable} \quad / \quad \varrho_{\varphi,\Omega}(u) < +\infty\}$$

is called the Musielak-Orlicz class (the generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$; equivalently

$$L_\varphi(\Omega) = \Big\{u : \Omega \longmapsto I\!R \quad \text{measurable} \quad / \quad \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \quad \text{for some } \lambda > 0\Big\}.$$

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\Big\{\lambda > 0 \quad / \quad \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\, dx \le 1\Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \le 1} \int_\Omega |u(x)v(x)|\, dx,$$

where $\psi(x,t)$ is the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. These two norms are equivalent [19].

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [19].

We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ has the Δ_2-condition for large values of t, or for all values of t, according to whether Ω has finite measure or not.

2.3 Musielak-Orlicz-Sobolev space

We now turn to the Musielak-Orlicz-Sobolev space. $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all measurable functions u such that u and its distributional derivatives up to order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$). Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives.

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{\varphi,\Omega}(D^\alpha u) \quad and \quad ||u||_{1,\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

for $u \in W^1L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively, and the pair $\langle W^1L_\varphi(\Omega), ||u||_{1,\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [19]:

$$\text{there exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x\in\Omega} \varphi(x, 1) \geq c.$$

The spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of the product of $n+1$ copies of $L_\varphi(\Omega)$. Denoting this product by ΠL_φ, we will use the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

The space $W_0^1E_\varphi(\Omega)$ is defined as the (norm) closure of the Schwartz space D(Ω) in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of D(Ω) in $W^1L_\varphi(\Omega)$. The following spaces of distribution will also be used :

$$W^{-1}L_\psi(\Omega) = \{f \in D'(\Omega); \quad f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \quad with \quad f_\alpha \in L_\psi(\Omega)\}$$

and

$$W^{-1}E_\psi(\Omega) = \{f \in D'(\Omega); \quad f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \quad with \quad f_\alpha \in E_\psi(\Omega)\}$$

If $\psi(x,t)$ has the Δ_2- condition, then the space $D(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the topology $\sigma(\Pi L_\varphi(\Omega), \Pi L_\psi(\Omega)$.

If $\psi(x,t)$ has the Δ_2- condition, then the space D(Ω) is dense in $W_0^1L_\varphi(\Omega)$ for the topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$ (see corollary 1 of [12]).

3 Essential assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, and $\varphi(x,t)$ be a Musielak-Orlicz function.
We set $\psi(x,t)$ the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$, we assume here that $\psi(x,t)$ satisfying the Δ_2-condition near infinity, then $L_\psi(\Omega) = E_\psi(\Omega)$.
Let $\gamma(x,t)$ be a Musielak-Orlicz function such that $\gamma \prec\prec \varphi$, and there exists a Orlicz function $M(t)$ such that

$$M(t) \leq ess \inf_{x\in\Omega} \varphi(x,t) \qquad \text{a.e. in} \quad \Omega. \tag{3.1}$$

We consider a Leray-Lions operator $A : D(A) \subset W_0^1L^\varphi(\Omega) \to W^{-1}L^\psi(\Omega)$ (not defined everywhere) given by

$$A(u) = -\text{div } a(x, u, \nabla u)$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R$ is a *Carathéodory* function (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$, and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω) which satisfies the following conditions

$$|a(x,s,\xi)| \leq K(x) + k_1\psi_x^{-1}(\gamma(x, k_2|s|)) + k_3\psi_x^{-1}(\varphi(x, k_4|\xi|)), \tag{3.2}$$

$$\big(a(x,s,\xi) - a(x,s,\xi^*)\big) \cdot \big(\xi - \xi^*\big) > 0 \qquad \text{for} \quad \xi \neq \xi^*, \tag{3.3}$$

$$a(x,s,\xi) \cdot \xi \geq \alpha \cdot \varphi(x, \frac{|\,\xi\,|}{\lambda}), \tag{3.4}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in I\!R \times I\!R^N$, where $K(x)$ is a nonnegative function lying in $E_\psi(\Omega)$ and $\alpha, \lambda > 0$ and $k_1, k_2, k_3, k_4 \geq 0$.

The nonlinear terms $g(x,s,\xi)$ is a *Carathéodory* functions satisfying

$$|g(x,s,\xi)| \leq c(x) + b(|s|)\varphi(x, \frac{|\xi|}{\lambda}), \tag{3.5}$$

where $b(\cdot) : I\!R^+ \longmapsto I\!R^+$ is a continuous and non-decreasing function such that $b(\cdot) \in L^1(I\!R)$. The nonnegative function $c(x) \in L^1(\Omega)$ and $\lambda > 0$.

We consider the problem

$$-\text{div}\, a(x,u,\nabla u) + g(x,u,\nabla u) = f \quad \text{in} \quad \Omega, \tag{3.6}$$

with $f \in L^1(\Omega)$.

4 Some technical Lemmas

We present here some lemmas, which will be used later in order to prove the existence theorem:

Lemma 4.1 *[8] Let Ω be an open subset of $I\!R^N$ with finite measure. Let φ, ψ and γ be Musielak functions such that $\gamma \prec\prec \psi$, and let $f : \Omega \times I\!R \to I\!R$ be a Carathéodory function such for a.e. $x \in \Omega$ and all $s \in I\!R$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|) \tag{4.1}$$

Lemma 4.2 *Let (f_n), $f \subset L^1(\Omega)$ such that $f_n \geq 0$ a.e. in Ω and $f_n \to f$ a.e. in Ω, with*

$$\int_\Omega f_n\, dx \longrightarrow \int_\Omega f\, dx.$$

Then $f_n \to f$ strongly in $L^1(\Omega)$.

Lemma 4.3 *Assuming that $(3.2)-(3.4)$ holds, and let $(u_n)_n$ be a sequence in $W_0^1 L_\varphi(\Omega)$ such that*

(i) *$u_n \rightharpoonup u$ weakly in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$,*

(ii) *$(a(x,u_n,\nabla u_n))_n$ is bounded in $(L_\psi(\Omega))^N$,*

(iii) *Let $\Omega_s = \{x \in \Omega, \quad |\nabla u| \leq s\}$ and χ_s his characteristic function, with*

$$\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx \longrightarrow 0 \quad as \quad n,s \to \infty, \tag{4.2}$$

then $\varphi(x, |\nabla u_n|) \longrightarrow \varphi(x, |\nabla u|)$ in $L^1(\Omega)$ for a subsequence.

Proof

Taking $s \geq r > 0$, we have:

$$\begin{aligned} 0 &\leq \int_{\Omega_r} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\ &\leq \int_{\Omega_s} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\ &= \int_{\Omega_s} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx \\ &\leq \int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx. \end{aligned} \tag{4.3}$$

thanks to (4.2), we obtain

$$\lim_{n\to\infty}\int_{\Omega_r}(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))\cdot(\nabla u_n-\nabla u)\,dx=0. \tag{4.4}$$

Using the same argument as in (3.5), we claim that,

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega. \tag{4.5}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx &= \int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi_s))\cdot(\nabla u_n-\nabla u\chi_s)\,dx\\ &+\int_\Omega a(x,u_n,\nabla u\chi_s)\cdot(\nabla u_n-\nabla u\chi_s)\,dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u\chi_s\,dx.\end{aligned} \tag{4.6}$$

For the second term on the right hand side of (4.6), we have $L_\psi(\Omega)=E_\psi(\Omega)$, $a(x,u_n,\nabla u\chi_s)\to a(x,u,\nabla u\chi_s)$ in $(E_\psi(\Omega))^N$ for the modular convergence, and $\nabla u_n \rightharpoonup \nabla u$ in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$, then

$$\begin{aligned}\lim_{s,n\to\infty}\int_\Omega a(x,u_n,\nabla u\chi_s)\cdot(\nabla u_n-\nabla u\chi_s)\,dx &= \lim_{s\to\infty}\int_\Omega a(x,u,\nabla u\chi_s)\cdot(\nabla u-\nabla u\chi_s)\,dx\\ &= \lim_{s\to\infty}\int_{\Omega/\Omega_s} a(x,u,0)\cdot\nabla u\,dx=0.\end{aligned} \tag{4.7}$$

Concerning the last term on the right hand of (4.6), since $(a(x,u_n,\nabla u_n))_n$ is bounded in $(E_\psi(\Omega))^N$ and using (4.5), we obtain

$$a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u) \quad \text{weakly in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega),\Pi L_\varphi(\Omega)),$$

which implies that

$$\begin{aligned}\lim_{s,n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u\chi_s\,dx &= \lim_{s\to\infty}\int_\Omega a(x,u,\nabla u)\cdot\nabla u\chi_s\,dx\\ &= \int_\Omega a(x,u,\nabla u)\cdot\nabla u\,dx.\end{aligned} \tag{4.8}$$

By combining (4.6) − (4.8) and thanks to (4.2), we conclude that

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \longrightarrow \int_\Omega a(x,u,\nabla u)\cdot\nabla u\,dx \quad \text{as} \quad n\to\infty. \tag{4.9}$$

On the other hand, we have $\varphi(x,\frac{|\nabla u_n|}{\lambda})\geq 0$ and $\varphi(x,\frac{|\nabla u_n|}{\lambda})\to\varphi(x,\frac{|\nabla u|}{\lambda})$ a.e. in Ω. Using the Fatou's Lemma, we obtain

$$\int_\Omega \varphi(x,\frac{|\nabla u|}{\lambda})\,dx \leq \liminf_{n\to\infty}\int_\Omega \varphi(x,\frac{|\nabla u_n|}{\lambda})\,dx. \tag{4.10}$$

Moreover, since $a(x,u_n,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,\frac{|\nabla u_n|}{\lambda})\geq 0$ and

$$a(x,u_n,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,\frac{|\nabla u_n|}{\lambda}) \longrightarrow a(x,u,\nabla u)\cdot\nabla u-\alpha\varphi(x,\frac{|\nabla u|}{\lambda}) \qquad \text{a.e. in} \quad \Omega,$$

In view of Fatou's Lemma, we get

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla u-\alpha\varphi(x,\frac{|\nabla u|}{\lambda})\,dx \leq \liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,\frac{|\nabla u_n|}{\lambda})\,dx,$$

using (4.9), we obtain

$$\int_\Omega \varphi(x, \frac{|\nabla u|}{\lambda})\, dx \geq \limsup_{n\to\infty} \int_\Omega \varphi(x, \frac{|\nabla u_n|}{\lambda})\, dx. \tag{4.11}$$

By combining (4.10) and (4.11), we deduce

$$\int_\Omega \varphi(x, \frac{|\nabla u_n|}{\lambda})\, dx \longrightarrow \int_\Omega \varphi(x, \frac{|\nabla u|}{\lambda})\, dx \qquad \text{as} \quad n \to \infty, \tag{4.12}$$

Using the Lemma 4.2, we conclude that

$$\varphi(x, \frac{|\nabla u_n|}{\lambda}) \longrightarrow \varphi(x, \frac{|\nabla u|}{\lambda}) \qquad \text{in} \quad L^1(\Omega), \tag{4.13}$$

which finishes our proof.

5 Main results

Let $k > 0$, we define the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Definition 5.1 *A measurable function* u *is called an entropy solution of the strongly nonlinear problem* (3.6) *if*

$$T_k(u) \in W_0^1 L_\varphi(\Omega), \qquad g(x, u, \nabla u) \in L^1(\Omega),$$

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \nu)\, dx + \int_\Omega g(x, u, \nabla u) T_k(u - \nu)\, dx = \int_\Omega f T_k(u - \nu)\, dx, \tag{5.1}$$

for any $\nu \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$,

Theorem 5.1 *Assuming that* (3.2) – (3.5) *holds and* $f \in L^1(\Omega)$, *then the problem* (3.6) *has at least one entropy solution.*

Proof of the Theorem 5.1.

Step 1 : Approximate problems.

Let $(f_n)_{n\in I\!N} \in W^{-1}E_\psi(\Omega) \cap L^\infty(\Omega)$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$-\text{div } a(x, u_n, \nabla u_n) + g_n(x, u_n, \nabla u_n) = f_n, \tag{5.2}$$

where

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|} \qquad \text{for} \quad n \in I\!N^*.$$

Note that

$$|g_n(x, s, \xi)| \leq |g(x, s, \xi)| \qquad \text{and} \qquad |g_n(x, s, \xi)| \leq n.$$

Since $g_n(x, s, \xi)$ is bounded, there exists at least one weak solution $u_n \in W_0^1 L_\varphi(\Omega)$ of equation (5.2) (see. Theorem 5 of [12]).

Step 2 : A priori estimates.

We define

$$B(s) = \frac{1}{\alpha}\int_0^s b(|\tau|)\,d\tau.$$

Note that, since the function $b(\cdot)$ is integrable on $I\!R$, then $0 \le B(\infty) := \frac{1}{\alpha}\int_0^{+\infty} b(|t|)\,dt$ is finite. Thus, taking $T_k(u_n)e^{B(|u_n|)} \in W_0^1L_\varphi(\Omega)$ as a test function in (5.2), we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)e^{B(|u_n|)}\,dx + \frac{1}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}\,dx\\&+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)e^{B(|u_n|)}\,dx = \int_\Omega f_nT_k(u_n)e^{B(|u_n|)}\,dx,\end{aligned}$$

Thanks to (3.4) and (3.5), we obtain

$$\begin{aligned}&\alpha\int_{\{|u_n|\le k\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})e^{B(|u_n|)}\,dx + \int_\Omega \varphi(x,\frac{|\nabla u_n|}{\lambda})b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}\,dx\\&\le \int_\Omega |c(x)||T_k(u_n)|e^{B(|u_n|)}\,dx + \int_\Omega b(|u_n|)\varphi(x,\frac{|\nabla u_n|}{\lambda})|T_k(u_n)|e^{B(|u_n|)}\,dx + \int_\Omega f_nT_k(u_n)e^{B(|u_n|)}\,dx,\end{aligned}$$

it follows that

$$\alpha\int_{\{|u_n|\le k\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})e^{B(|u_n|)}\,dx \le ke^{B(\infty)}\int_\Omega |c(x)|\,dx + ke^{B(\infty)}\int_\Omega |f(x)|\,dx.$$

Then, there exists a constant C_1 that does not depend on k and n such that

$$\int_\Omega \varphi(x,\frac{|\nabla T_k(u_n)|}{\lambda})\,dx \le kC_1. \tag{5.3}$$

Thus $(T_k(u_n))_n$ is bounded in $W_0^1L_\varphi(\Omega)$, uniformly in n, then there exists a subsequence still denoted $(T_k(u_n))_{n\in I\!N}$ and $v_k \in W_0^1L_\varphi(\Omega)$ such that

$$T_k(u_n) \rightharpoonup v_k \quad \text{weakly in} \quad W_0^1L_\varphi(\Omega) \quad \text{for} \quad \sigma(\Pi L_\varphi,\Pi E_\psi) = \sigma(\Pi L_\varphi,\Pi L_\psi) \tag{5.4}$$

then

$$T_k(u_n) \to v_k \quad \text{strongly in} \quad E_\varphi(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega. \tag{5.5}$$

Step 3 : Convergence in measure of u_n

Thanks to (3.1), we have

$$M(t) \le ess\inf_{x\in\Omega}\varphi(x,t) \qquad \text{with} \qquad \lim_{t\to 0}\frac{M(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{M(t)}{t} = \infty.$$

Then, $\varphi(x,t)$ dominate $M(t)$ near infinity. By Lemma 5.7 of [16], there exists two positive constants C_2 and C_3, and a function $q(\cdot) \in L^1(\Omega)$ such that

$$C_2\int_\Omega M(|T_k(u_n)|)\,dx + \int_\Omega q(x)\,dx \le \int_\Omega M(C_3\frac{|\nabla T_k(u_n)|}{\lambda}) + q(x)\,dx \le \int_\Omega \varphi(x,\frac{|\nabla T_k(u_n)|}{\lambda})\,dx.$$

So, in virtue of (5.3), we obtain

$$\int_\Omega M(|T_k(u_n)|)\,dx \le kC_4 \qquad \text{for} \quad k \ge 1. \tag{5.6}$$

https://doi.org/10.54582/TSJ.2.2.18

Then, we deduce that,

$$\begin{aligned} M(k)\ \text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} M(|T_k(u_n)|)\,dx \\ &\leq \int_\Omega M(|T_k(u_n)|)\,dx \\ &\leq kC_4. \end{aligned}$$

Hence,

$$\text{meas}(\{|u_n| > k\}) = \frac{kC_4}{M(k)} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{5.7}$$

For all $\delta > 0$, we have

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.$$

Let $\varepsilon > 0$, using (5.7) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{5.8}$$

Moreover, in view of (5.5) we have $T_k(u_n) \to v_k$ strongly in $E_\varphi(\Omega)$, then, we can assume that $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure. Thus,

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(\delta, \varepsilon). \tag{5.9}$$

By combining (5.8) – (5.9), we conclude that

$$\forall \delta, \varepsilon > 0 \quad \text{there exists} \quad n_0 = n_0(\delta, \varepsilon) \quad \text{such that} \quad \text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \quad \forall n, m \geq n_0(\delta, \varepsilon),$$

it follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^1 L_\varphi(\Omega) \quad \text{for} \quad \sigma(\Pi L_\varphi, \Pi E_\psi)$$

it follows that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad E_\varphi(\Omega). \tag{5.10}$$

Step 4 : Strong convergence of truncations.

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots$ various real-valued functions of real variables that converge to 0 as n tends to infinity.
For $h > k > 0$, we set

$$b_k := \max\{b(s) : |s| \leq k\}$$

Let $\phi_k(s) = s\exp(\gamma s^2)$ with $\gamma = \left(\frac{b_k}{\alpha}\right)^2$, it's clear that

$$\phi_k'(s) - \frac{2b_k}{\alpha}|\phi_k(s)| \geq \frac{1}{2} \quad \forall s \in I\!R.$$

Taking $M = 4k + h$, we define

$$z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u) \qquad \text{and} \qquad \omega_n := T_{2k}(z_n).$$

Using $\phi_k(\omega_n)e^{B(|u_n|)} \in W_0^1 L_\varphi(\Omega)$ as a test function in (5.2), we obtain

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \omega_n \phi_k'(\omega_n) e^{B(|u_n|)}\,dx + \frac{1}{\alpha}\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n \phi_k(\omega_n) b(|u_n|)\text{sign}(u_n) e^{B(|u_n|)}\,dx \\ &+ \int_\Omega H_n(x, u_n, \nabla u_n)\phi_k(\omega_n) e^{B(|u_n|)}\,dx = \int_\Omega f_n \phi_k(\omega_n) e^{B(|u_n|)}\,dx. \end{aligned}$$

Since $\phi_k(\omega_n)$ have the same sign as u_n on the set $\{|u_n| > k\}$, then

$$\begin{aligned}
&\frac{1}{\alpha}\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\phi_k(\omega_n)b(|u_n|)\text{sign}(u_n)e^{B(|u_n|)}\,dx\\
&=\frac{1}{\alpha}\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n b(|u_n|)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\geq\int_{\{|u_n|>k\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})b(|u_n|)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx,
\end{aligned}$$

Also, we have $\nabla\omega_n = 0$ on $\{|u_n| \geq M\}$ and using (3.5) we conclude that

$$\begin{aligned}
&\int_{\{|u_n|\leq M\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla\omega_n\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&-\int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\frac{b(|u_n|)}{\alpha}|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&-\int_{\{|u_n|\leq k\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})b(|u_n|)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\leq e^{B(\infty)}\int_\Omega(|f_n(x)|+|c(x)|)|\phi_k(\omega_n)|\,dx.
\end{aligned}$$

We have $\omega_n = T_k(u_n) - T_k(u)$ on $\{|u_n| \leq k\}$, we conclude that

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{k<|u_n|\leq M\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla\omega_n\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\leq e^{B(\infty)}\int_\Omega(|f_n(x)|+|c(x)|)|\phi_k(\omega_n)|\,dx.
\end{aligned}\tag{5.11}$$

Concerning the second term on the left-hand side of (5.11), we have

$$\begin{aligned}
&\int_{\{k<|u_n|\leq M\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla\omega_n\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&=\int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla z_n\;\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\geq -e^{B(\infty)}\phi_k'(2k)\int_{\{k<|u_n|\leq M\}}|a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|\,dx,
\end{aligned}$$

We have $\nabla T_k(u) \in (L_\varphi(\Omega))^N$, and since $(|a(x,T_M(u_n),\nabla T_M(u_n))|)_n$ is bounded in $L_\psi(\Omega) = E_\psi(\Omega)$, there exists $\zeta \in E_\psi(\Omega)$ such that $|a(x,T_M(u_n),\nabla T_M(u_n))| \rightharpoonup \zeta$ weakly in $E_\psi(\Omega)$ for $\sigma(E_\psi(\Omega), L_\varphi(\Omega))$. Therefore,

$$\int_{\{k<|u_n|\leq M\}}|a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|\,dx\longrightarrow\int_{\{k<|u|\leq M\}}\zeta\,|\nabla T_k(u)|\,dx=0.\tag{5.12}$$

It follows that

$$\int_{\{k<|u_n|\leq M\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla\omega_n\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\geq\varepsilon_1(n).\tag{5.13}$$

Then, using (5.11), we deduce that

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\leq e^{B(\infty)}\int_\Omega(|f_n(x)|+|c(x)|)|\phi_k(\omega_n)|\,dx+\varepsilon_2(n).
\end{aligned}\tag{5.14}$$

Now, we will study each terms on the left-hand side of (5.14).

First estimate :

We define $\Omega_s = \{x \in \Omega : |\nabla T_k(u(x))| \leq s\}$ and denote by χ_s the characteristic function of Ω_s. For the first term on the left-hand side of (5.14), we have

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u)\chi_s-\nabla T_k(u))\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&=\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad-\int_{\Omega\setminus\Omega_s} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx.
\end{aligned}\tag{5.15}$$

For the second term on the right hand side of (5.15), we have $a(x,T_k(u_n),\nabla T_k(u)\chi_s)\to a(x,T_k(u),\nabla T_k(u)\chi_s)$ in $(E_\psi(\Omega))^N$, and since $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$, then

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad=\int_\Omega a(x,T_k(u),\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u)-\nabla T_k(u)\chi_s)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx\\
&\quad=\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\cdot\nabla T_k(u)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx.
\end{aligned}\tag{5.16}$$

Concerning the third term on the right hand side of (5.15), since $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(E_\psi(\Omega))^N$, there exists $\eta\in(E_\psi(\Omega))^N$ such that $a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\eta$ weakly in $(E_\psi(\Omega))^N$ for $\sigma(\Pi E_\psi(\Omega),\Pi L_\varphi(\Omega))$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega\setminus\Omega_s} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad=\int_{\Omega\setminus\Omega_s}\eta\cdot\nabla T_k(u)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx.
\end{aligned}\tag{5.17}$$

For the last term of (5.15), we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad=\int_{\{|u|>k\}}\eta\cdot\nabla T_k(u)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx=0.
\end{aligned}\tag{5.18}$$

By combining (5.16) − (5.18), we deduce that

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&=\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\phi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\Omega\setminus\Omega_s}\big(a(x,T_k(u),0)-\eta\big)\cdot\nabla T_k(u)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx+\varepsilon_3(n)
\end{aligned}\tag{5.19}$$

Second estimate :

For the second term on the right-hand side of (5.14), we have

$$\begin{aligned}&\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&=\frac{2b_k}{\alpha}\int_\Omega\big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&+\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&+\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\chi_s|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx.\end{aligned}\tag{5.20}$$

For the second term on the right-hand side of (5.20), we have $h>k>0$, similarly to (5.16) we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&\quad=\int_\Omega a(x,T_k(u),\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u)-\nabla T_k(u)\chi_s)|\phi_k(T_{2k}(u-T_h(u)))|e^{B(|u|)}\,dx\\&\quad=\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\cdot\nabla T_k(u)|\phi_k(T_{2k}(u-T_h(u)))|e^{B(|u|)}\,dx=0.\end{aligned}\tag{5.21}$$

Concerning the third term on the right-hand side of (5.20), we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\chi_s|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&\quad=\int_\Omega\eta\cdot\nabla T_k(u)\chi_s|\phi_k(T_{2k}(u-T_h(u)))|e^{B(|u|)}\,dx=0.\end{aligned}\tag{5.22}$$

It follows that

$$\begin{aligned}&\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&=\frac{2b_k}{\alpha}\int_\Omega\big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)|\phi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&\quad+\varepsilon_4(n).\end{aligned}\tag{5.23}$$

By combining (5.14), (5.19) and (5.23), we conclude that

$$\begin{aligned}&\frac12\int_\Omega\big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)\,dx\\&\le\int_\Omega\big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big)\\&\qquad\times\big(\phi_k'(\omega_n)-\frac{2b_k}{\alpha}|\phi_k(\omega_n)|\big)e^{B(|u_n|)}\,dx\\&\le\int_{\Omega\setminus\Omega_s}\big(a(x,T_k(u),0)-\eta\big)\cdot\nabla T_k(u)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx\\&\qquad+e^{B(\infty)}\int_\Omega(|f(x)|+|c(x)|)|\phi_k(T_{2k}(u-T_h(u)))|\,dx+\varepsilon_5(n).\end{aligned}\tag{5.24}$$

We have $|\phi_k(T_{2k}(u-T_h(u)))|\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, then

$$\int_\Omega(|f(x)|+|c(x)|)|\phi_k(T_{2k}(u-T_h(u)))|\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty,$$

and since $\big(a(x,T_k(u),0)-\eta\big)\cdot\nabla T_k(u)\in L^1(\Omega)$, then

$$\int_{\Omega\setminus\Omega_s}\big(a(x,T_k(u),0)-\eta\big)\cdot\nabla T_k(u)\phi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx\longrightarrow 0\qquad\text{as}\quad s\to\infty.$$

Therefore, we conclude that

$$\lim_{n,s\to\infty}\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\,dx = 0. \quad (5.25)$$

In view of Lemma 4.3, we deduce that

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega, \quad (5.26)$$

and

$$\varphi(x,\frac{|\nabla T_k(u_n)|}{\lambda}) \longrightarrow \varphi(x,\frac{|\nabla T_k(u)|}{\lambda}) \qquad \text{in} \quad L^1(\Omega). \quad (5.27)$$

Step 4 : The equi-integrability of $g_n(x,u_n,\nabla u_n)$

In order to pass to the limit in the approximate problem, we shall show that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{in} \quad L^1(\Omega). \quad (5.28)$$

Thanks to Vitali's theorem, it suffices to prove that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. We set

$$\overline{B}(s) = \frac{2}{\alpha}\int_0^s b(|\tau|)\,d\tau.$$

By taking $T_1(u_n - T_h(u_n))e^{\overline{B}(|u_n|)}$ as a test function in (5.2), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_1(u_n - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\ &+\frac{2}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n b(|u_n|)|T_1(u_n - T_h(u_n))|e^{\overline{B}(|u_n|)}\,dx\\ &+\int_\Omega g_n(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\ &=\int_\Omega f_nT_1(u_n - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx.\end{aligned}$$

According to (3.4) and (3.5), we obtain

$$\begin{aligned}&\alpha\int_{\{h\le|u_n|<h+1\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})e^{\overline{B}(|u_n|)}\,dx + 2\int_{\{h\le|u_n|\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})b(|u_n|)|T_1(u_n - T_h(u_n))|e^{\overline{B}(|u_n|)}\,dx\\ &\le\int_{\{h\le|u_n|\}}(|f_n(x)|+|c(x)|)e^{\overline{B}(|u_n|)}\,dx + \int_{\{h\le|u_n|\}}\varphi(x,\frac{|\nabla u_n|}{\lambda})|T_1(u_n - T_h(u_n))|b(|u_n|)e^{\overline{B}(|u_n|)}\,dx,\end{aligned} \quad (5.29)$$

it follows that

$$\int_{\{h+1\le|u_n|\}} b(|u_n|)\varphi(x,\frac{|\nabla u_n|}{\lambda})\,dx \le e^{\overline{B}(\infty)}\int_{\{h\le|u_n|\}}(|f(x)|+|c(x)|)\,dx.$$

Thus, for all $\eta>0$, there exists $h(\eta)\ge 1$ such that

$$\int_{\{h(\eta)\le|u_n|\}} b(|u_n|)\varphi(x,\frac{|\nabla u_n|}{\lambda})\,dx \le \frac{\eta}{2}. \quad (5.30)$$

On the other hand, we set

$$b_{h(\eta)} := \max\{b(s) : |s|\le h(\eta)\}.$$

For any measurable subset $E\subseteq\Omega$, we have

$$\int_E b(|u_n|)\varphi(x,\frac{|\nabla u_n|}{\lambda})\,dx \le b_{h(\eta)}\int_E\varphi(x,\frac{|\nabla T_{h(\eta)}(u_n)|}{\lambda})\,dx + \int_{\{h(\eta)\le|u_n|\}} b(|u_n|)\varphi(x,\frac{|\nabla u_n|}{\lambda})\,dx. \quad (5.31)$$

From (5.27), there exists $\delta(\eta) > 0$ such that, for any meas$(E) \leq \delta(\eta)$ we have

$$b_{h(\eta)} \int_E \varphi(x, \frac{|\nabla T_{h(\eta)}(u_n)|}{\lambda})\, dx \leq \frac{\eta}{2}. \tag{5.32}$$

Finally, by combining (5.30), (5.31) and (5.32), one easily has

$$\int_E b(|u_n|)\varphi(x, \frac{|\nabla u_n|}{\lambda})\, dx \leq \eta \quad \text{for all} \quad \text{meas}(E) \leq \delta(\eta). \tag{5.33}$$

Using (5.12), we deduce that $(g_n(x, u_n, \nabla u_n))_n$ is equi-integrable, and since

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{a.e. in} \quad \Omega,$$

we conclude (5.28).

Step 5 : Passage to the limit

Let $\nu \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, taking $T_k(u_n - \nu)$ as a test function in (5.2), we get

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - \nu)\, dx + \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - \nu)\, dx = \int_\Omega f_n T_k(u_n - \nu)\, dx. \tag{5.34}$$

Choosing $M = k + \|\nu\|_\infty$, if $|u_n| > M$ then $|u_n - \nu| \geq |u_n| - \|\nu\|_\infty > k$, therefore $\{|u_n - \nu| \leq k\} \subseteq \{|u_n| \leq M\}$. Firstly, we can write the first term on the left-hand side of the above relation as

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - \nu)\, dx &= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot (\nabla T_M(u_n) - \nabla \nu)\chi_{\{|u_n - \nu| \leq k\}}\, dx \\ = \int_\Omega (a(x, T_M(u_n), \nabla T_M(u_n)) &- a(x, T_M(u_n), \nabla \nu)) \cdot (\nabla T_M(u_n) - \nabla \nu)\chi_{\{|u_n - \nu| \leq k\}}\, dx \\ + \int_\Omega a(x, T_M(u_n), \nabla \nu) &\cdot (\nabla T_M(u_n) - \nabla \nu)\chi_{\{|u_n - \nu| \leq k\}}\, dx.\end{aligned} \tag{5.35}$$

We have

$$\begin{aligned}&(a(x, T_M(u_n), \nabla T_M(u_n)) - a(x, T_M(u_n), \nabla \nu)) \cdot (\nabla T_M(u_n) - \nabla \nu)\chi_{\{|u_n - \nu| \leq k\}} \\ &\longrightarrow (a(x, T_M(u), \nabla T_M(u)) - a(x, T_M(u), \nabla \nu)) \cdot (\nabla T_M(u) - \nabla \nu)\chi_{\{|u - \nu| \leq k\}} \quad \text{a.e. in} \quad \Omega.\end{aligned} \tag{5.36}$$

According to (3.3) and Fatou's lemma, we obtain

$$\begin{aligned}\liminf_{n\to\infty} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - \nu)\, dx \\ &\geq \int_\Omega (a(x, T_M(u), \nabla T_M(u)) - a(x, T_M(u), \nabla \nu)) \cdot (\nabla T_M(u) - \nabla \nu)\chi_{\{|u - \nu| \leq k\}}\, dx \\ &\quad + \lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla \nu) \cdot (\nabla T_M(u_n) - \nabla \nu)\chi_{\{|u_n - \nu| \leq k\}}\, dx.\end{aligned} \tag{5.37}$$

For the second term on the right-hand side of (5.37), we have $a(x, T_M(u_n), \nabla \nu) \to a(x, T_M(u), \nabla \nu)$ in $(E_\psi(\Omega))^N$, and $\nabla T_M(u_n) \rightharpoonup \nabla T_M(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$, then

$$\begin{aligned}\lim_{n\to\infty} &\int_\Omega a(x, T_M(u_n), \nabla \nu) \cdot (\nabla T_M(u_n) - \nabla \nu)\chi_{\{|u_n - \nu| \leq k\}}\, dx \\ &= \int_\Omega a(x, T_M(u), \nabla \nu) \cdot (\nabla T_M(u) - \nabla \nu)\chi_{\{|u - \nu| \leq k\}}\, dx.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - \nu)\, dx &\geq \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot (\nabla T_M(u) - \nabla \nu)\chi_{\{|u - \nu| \leq k\}}\, dx \\ &= \int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \nu)\, dx.\end{aligned}$$

On the other hand, being $T_k(u_n-\nu) \rightharpoonup T_k(u-\nu)$ weak-$\star$ in $L^\infty(\Omega)$ and thanks to (5.28), we deduce that

$$\int_\Omega g_n(x,u_n,\nabla u_n)\ T_k(u_n-\nu)\,dx \longrightarrow \int_\Omega g(x,u,\nabla u)\ T_k(u-\nu)\,dx \tag{5.38}$$

and

$$\int_\Omega f_n T_k(u_n-\nu)\,dx \longrightarrow \int_\Omega f\ T_k(u-\nu)\,dx. \tag{5.39}$$

Hence, putting all the terms together, we conclude the proof of Theorem 5.1.

Exemple 5.1 *Taking* $\varphi(x,t)=|t|^{p(x)}$ *for* $1<p(x)<\infty$. *Let* $f\in L^1(\Omega)$, *we consider the following Carathéodory functions*

$$a(x,u,\nabla u)=|\nabla u|^{p(x)-2}\nabla u \qquad \textit{and} \qquad H(x,u,\nabla u)=e^{-|u|^2}|\nabla u|^{p(x)}.$$

It is clear that $a(x,u,\nabla u)$ *and* $g(x,u,\nabla u)$ *verifies* $(3.2)-(3.4)$ *and* (3.5) *respectively. In view of the Theorem 5.1, the problem*

$$\begin{cases} -div\big(|\nabla u|^{p(x)-2}\nabla u\big)+e^{-|u|^2}|\nabla u|^{p(x)}=f & in\ \ \Omega,\\ u\ =0 & on\ \ \partial\Omega, \end{cases} \tag{5.40}$$

has at least one entropy solution, i.e.

$$T_k(u)\in W_0^{1,p(x)}(\Omega) \qquad \textit{and} \qquad e^{-|u|^2}|\nabla u|^{p(x)}\in L^1(\Omega),$$

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla T_k(u_n-\nu)\,dx+\int_\Omega e^{-|u|^2}|\nabla u|^{p(x)}T_k(u_n-\nu)\,dx=\int_\Omega fT_k(u_n-\nu)\,dx, \tag{5.41}$$

for any $\nu\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$.

References

[1] R. A. Adams and J. F. Fournier; *Sobolev spaces*,2nd edition, Pure and Applied Mathematics, 140, Academic Press, Amsterdam, 2003.

[2] E. Azroul, A. Benkirane and M. Rhoudaf, *On some strongly nonlinear elliptic problems in L1-data with a nonlinearity having a constant sign in Orlicz spaces via penalization methods.* Aust. J. Math. Anal. Appl. 7 (2010), no. 1, Art. 5, 1-25.

[3] A. Aissaoui Fqayeh, A. Benkirane, M. El Moumni and A. Youssfi, Existence of renormalized solutions for some strongly nonlinear elliptic equations in Orlicz spaces, Georgian Math. J. 22(3) (2015), 305-321.

[4] B. Al Hamzah, N. Yebari, Global Existence in Reaction Diffusion Nonlinear Parabolic Partail Differential Equation in Image Processing, Global Journal of Advanced Engineering Technologies and Sciences. 3(5): May, 2016 , 42-55.

[5] B. Al Hamzah, N. Yebari, Global Existence for Reaction-Diffusion Systems Modeling Ions Electromigration Through Biological Membranes with Mass Control and Critical Growth with Respect to the Gradient. Topological Methods in Nonlinear Analysis, Vol. 53 No 1 (March 2019), 225-256.

[6] M. Al-Hawmi and M. AL-Hasisi, *Existence of renormalized solutions for nonlinear elliptic problems in Musielak-Orlicz-Sobolev spaces.* J. Math. Comput. Sci. 12(23)(2021), 1-31.

[7] M. Al-Hawmi, E.Azroul, H. Hjiaj and A.Touzani, Existence of entropy solutions for some anisotropic quasilinear elliptic unilateral problems, Afr. Mat. 28 (2017), 357-378.

[8] M.AL-Hawmi, A. Benkirane, H. Hjiaj and A. Touzani, Existence and Uniqueness of Entropy Solutions for some Nonlinear Elliptic Unilateral Problems in Musielak-Orlicz-Sobolev Spaces, Ann. Univ. Craiova, Math. Computer Sci. Ser. 44(1)(2017), 1-20.

[9] M. Al-Hawmi, Abdelmoujib Benkirane, Hassane Hjiaj and Abdelfattah Touzani, Existence of solutions for some nonlinear elliptic problems involving Mintys Lemma. Ricerche di Matematica A Journal of Pure and Applied Mathematics , (2018), 1-24.

[10] A. Benkirane and A. Elmahi; *An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces*, Nonlinear Analysis 36 (1999)11-24.

[11] A. Benkirane and A. Elmahi; *Almost everywhere convergence of gradients of solutions to elliptic equations in Orlicz spaces and application*, Nonlinear Analysis. T.M.A. 28(11)1769-1784.

[12] A. Benkirane and M. Sidi El Vally; *An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces.* Bull. Belg. Math. Soc. Simon Stevin 20 (2013), no. 1, 57-75.

[13] L. Boccardo, T. Gallouet and F. Murat;*A unified presentation of two existence results for problems with natural growth.* Progress in partial differential equations: the Metz surveys, 2 (1992), 127-137

[14] A. Bensoussan, L. Boccardo and F. Murat; *On a nonlinear partial differential equation having natural growth terms and unbounded solution*, Ann. Inst. Henri Poincar 5 (4)(1988), 347-364.

[15] A. Elmahi and D. Meskine; *Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces.* abstr. Appl. Anal. 2004, no.12, 1031-1045.

[16] J.P. Gossez; *Nonlinear elliptic boundary value prolems for equations with rapidly (or slowly) increasing coefficients*, Trans. Am. Math. Soc. 190 (1974), 163-205.

[17] J.P. Gossez and V. Mustonen; *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Anal. 11 (1987), 379-392.

[18] R. Landes, *On Galerkin's method in the existence theory for quasilinear elliptic equations*, J. funct. Analysis 39, 123-148 (1980).

[19] J. Musielak; *Modular spaces and Orlicz spaces*, Lecture Notes in Math. 1034 (1983).

https://doi.org/10.54582/TSJ.2.2.18