تعميم وايبول للتوزيع الأسي

Weibull Generalized Exponential Distribution

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ملخص البحث

تقدم هذه الورقة البحثية تعميم جديد للتوزيع الأسي، وهو عبارة عن نموذج أو موديل يحتوي على ثلاثة بارامترات، يسمى تعميم وايبول للتوزيع الأسي، والذي له دالة نسبة المخاطر على حياة أوبقاء أي نظام على شكل bathtub-shaped ، تم إيجاد بعض الخصائص الإحصائية للنموذج الجديد، مثل: مقاييس الموضع، والعزوم، والدالة المولدة لها، ودالة الموثوقية، والاحصاء المرتب. كما تم تقدير بارامترات النموذج باستخدام طريقة الاحتمال الأعظم ومشتقة مصفوفة فيشر. وأخيراً تم توضيح أهمية التعميم الجديد وكفاءته، من خلال تطبيق لبيانات حقيقية، ومقارنة مع تعميمات أخرى للتوزيع الأسي.

كلمات مفتاحية:

تعميم وايبول، التوزيع الأسي، تعميم التوزيع الأسي، تقدير البارامترات، طريقة الاحتمال الأعظم.

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Abstract

This paper introduces a new three-parameters model called the Weibull Generalized exponential distribution (WGED) which exhibits bathtub-shaped hazard rate. Some of its statistical properties are obtained including quantile, moments, generating function, reliability and order statistics. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is derived. We illustrate the usefulness of the proposed model by application to real data.

Keywords: Weibull-G class; exponential distribution; generalized exponential; estimating parameters; maximum likelihood estimation..

Weibull Generalized Exponential Distribution

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Abstract

This paper introduces a new three-parameters model called the Weibull Generalized exponential distribution which exhibits bathtub-shaped hazard rate. Some of its statistical properties are obtained including quantile, moments, generating function, reliability and order statistics. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is derived. We illustrate the usefulness of the proposed model by application to real data.

Keywords: *Weibull-Generalized class; exponential distribution; generalized exponential; maximum likelihood estimation.*

1 Introduction

The exponential distribution (ED), [3, 4], has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 1. The origin and other aspects of this distribution can be found in [4, 5, 6, 7]. A random variable X is said to have ED with parameters $\lambda > 0$ if it's probability density function (pdf) is given by

$$g(x) = \lambda e^{-\lambda x}, \quad x > 0, \tag{1.1}$$

while the cumulative distribution function (cdf) is given by

$$G(x) = 1 - e^{-\lambda x}, \quad x > 0. \tag{1.2}$$

The survival function is given by the equation

$$S(x) = 1 - G(x) = e^{-\lambda x}, \quad x > 0, \tag{1.3}$$

and the hazard function is

$$h(x) = \lambda. \tag{1.4}$$

Weibull distribution introduced by [22] is a popular distribution for modeling phenomenon with monotonic failure rates. But this distribution does not provide a good fit to data sets with bathtub shaped or upside-down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering and biological studies. Hence a number of new distributions modeling the data in a better way have been constructed in literature as ramifications of Weibull distribution. Bourguignon et al. [15] introduced and studied generality a family of univariate distributions with two additional parameters. Similarly as the extended Weibull, Gurvich et al. [9] and gamma families, Zografos and Balakrshnan [23], using the Weibull generator applied to the odds ratio $\frac{G(x)}{1-G(x)}$. If $G(x)$ is the baseline cumulative distribution function (cdf) of a random variable, with probability density function (pdf) $g(x)$ and the Weibull cumulative distribution function is

$$F(x;a,b)=1-e^{-ax^{b}},\quad x\geq 0, \tag{1.5}$$

with parameters a and b are positive. Based on this density, by replacing x with ratio $\frac{G(x)}{1-G(x)}$. The cdf of Weibull- generalized distribution, say Weibull-G distribution with two extra parameters a and b, is defined by, Bourguignon et al. [15]

$$\begin{aligned} F(x;a,b,\lambda) &= \int_0^{\frac{G(x;\lambda)}{1-G(x;\lambda)}} abt^{b-1}e^{-ax^{b}}dt \\ &= 1-e^{-a[\frac{G(x;\lambda)}{1-G(x;\lambda)}]^{b}},\quad x\geq 0,\quad a,b\geq 0. \end{aligned} \tag{1.6}$$

Where $G(x;\lambda)$ is a baseline cdf, which depends on a parameter λ. The corresponding family pdf becomes

$$f(x;a,b,\lambda)=ab\,g(x;\lambda)\frac{[G(x;\lambda)]^{b-1}}{[1-G(x;\lambda)]^{b+1}}e^{-a[\frac{G(x;\lambda)}{1-G(x;\lambda)}]^{b}}. \tag{1.7}$$

A random variable X with pdf Eq. (1.7) is denoted by X distributed weibll-G(a,b,λ), $x\in R$, $a,b>0$. the additional parameters induced by the weibull generator are sought as a manner to furnish a more flexible distribution. If $b=1$, it corresponds to the exponential- generator. An interpretation of the weibull-G family of distributions can by given as follows (Coorary, [2]) is a similar context.
Let Y be a lifetime random variable having a certain continuous G distribution. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $\frac{G(x)}{1-G(x)}$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Weibull model with scale a and shape b. We can write

$$Pr(Y\leq x)=Pr\left(X\leq\frac{G(x)}{1-G(x)}\right)=F(x;a,b,\lambda).$$

Which is given by Eq. (1.6). The survival function of the Weibull-G family is given by

$$R(x;a,b,\lambda)=1-F(x;a,b,\lambda)=e^{-a[\frac{G(x)}{1-G(x)}]^{b}}, \tag{1.8}$$

and hazard rate function of the Weibull-G family is given by

$$\begin{aligned} h(x;a,b,\lambda) &= \frac{f(x;a,b,\lambda)}{1-F(x;a,b,\lambda)}=\frac{ab\,g(x;\lambda)[G(x;\lambda)]^{b-1}}{[1-G(x;\lambda)]^{b+1}} \\ &= ab\,h(x;\lambda)\,\frac{[G(x;\lambda)]^{b-1}}{[1-G(x;\lambda)]^{b}}, \end{aligned} \tag{1.9}$$

where $h(x;\lambda) = \frac{g(x;\lambda)}{1-G(x;\lambda)}$. The multiplying quantity $\frac{ab\, g(x;\lambda)[G(x;\lambda)]^{b-1}}{[1-G(x;\lambda)]^{b}}$ works as a corrected factor for the hazard rate function of the baseline model Eq.(1.6) can deal with general situation in modeling survival data with various shapes of the hazard rate function. By using the power series for the exponential function, we obtain

$$e^{-a[\frac{G(x)}{1-G(x)}]^{b}} = \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \left(\frac{G(x;\lambda)}{1-G(x;\lambda)} \right)^{ib}, \tag{1.10}$$

substituting from Eq.(1.10) into Eq. (1.7), we get

$$f(x;a,b,\lambda) = ab\, g(x;\lambda) \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \frac{[G(x;\lambda)]^{b(i+1)-1}}{[1-G(x;\lambda)]^{b(i+1)+1}}. \tag{1.11}$$

Using the generalized binomial theorem we have

$$[1-G(x;\lambda)]^{-(b(i+1)+1)} = \sum_{j=0}^{\infty} \frac{\Gamma(b(i+1)+j+1)}{j!\Gamma(b(i+1)+1)} [G(x;\lambda)]^{j}. \tag{1.12}$$

Inserting Eq. (1.12) in Eq. (1.11), the Weibull-G family density function is

$$f(x;a,b,\lambda) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i a^{i+1} b\, \Gamma(b(i+1)+j+1)}{i!j!\Gamma(b(i+1)+1)} g(x;\lambda)[G(x;\lambda)]^{b(i+1)+j-1}. \tag{1.13}$$

For more details on Weibull-G family, we can refer [8, 10, 11, 14, 18]
In Section 2, we define the cumulative, density and hazard functions of the Weibull Generalized Exponential distribution (WGED) . In Sections 3 and 4, we introduced the statistical properties include, quantile function skewness and kurtosis, rth moments and moment generating function. The distribution of the order statistics is expressed in Section 5. Finally, maximum likelihood estimation of the parameters is determined in Section 6. Real data sets are analyzed in Section 7 and the results are compared with existing distributions. Finally we introduce the conclusions in Section 8.

2 The Weibull Generalized Exponential Distribution

In this Section, we study the three parameters WGED. Using $G(x)$ and $g(x)$ in Eq. (1.13) to be the cdf and pdf of Eq. (1.6) and Eq. (1.7). The cumulative distribution function (cdf) of the Weibull-G exponential distribution (WGED) is given by

$$F(x;a,b,\lambda) = 1 - e^{-a[e^{\lambda x}-1]^{b}}, \quad x>0, \quad a,b,\lambda>0. \tag{2.1}$$

The pdf corresponding to Eq. (2.1) is given by

$$f(x;a,b,\lambda) = ab\lambda\, e^{\lambda x}[e^{\lambda x}-1]^{b-1} e^{-a[e^{\lambda x}-1]^{b}}, \quad x>0, \tag{2.2}$$

where $a, b > 0$ and $\lambda > 0$ are two additional shape parameters.
Plots of the cdf, Eq. (2.1), of the WGED for some parameter values are displayed in Figure 1,

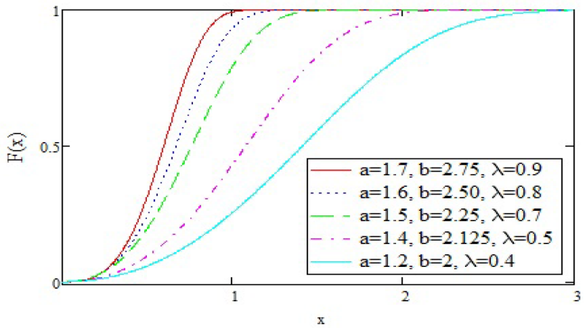

Figure 1: The cdf of the WGED.

We denote by $X \sim WGED(a, b, \lambda)$ a random variable having the pdf Eq. (2.1). The survival function, $S(x)$, hazard rate function, $h(x)$, reversed hazard rate function, $r(x)$ and cumulative hazard rate function $H(x)$ of X are given by

$$S(x; a, b, \lambda) = 1 - F(x; a, b, \lambda) = e^{-a[e^{\lambda x}-1]^b}, \quad x > 0, \tag{2.3}$$

$$h(x; a, b, \lambda) \;=\; ab\lambda\, e^{\lambda x}[e^{\lambda x} - 1]^{b-1}, \quad x > 0, \tag{2.4}$$

$$r(x; a, b, \lambda) \;=\; \frac{ab\lambda\, e^{\lambda x}[e^{\lambda x} - 1]^{b-1} \cdot e^{-a[e^{\lambda x}-1]^b}}{1 - e^{-a[e^{\lambda x}-1]^b}}, \quad x > 0, \tag{2.5}$$

and

$$H(x; a, b, \lambda) = \int_0^\infty h(x; a, b, \lambda)dx = a\left[e^{e^{\lambda x}} - 1\right]^b, \tag{2.6}$$

respectively. Plots of $S(x; a, b, \lambda), h(x; a, b, \lambda), r(x; a, b, \lambda)$ and $H(x; a, b, \lambda)$ of the WGED for some parameters values are displayed in Figures 2–6.

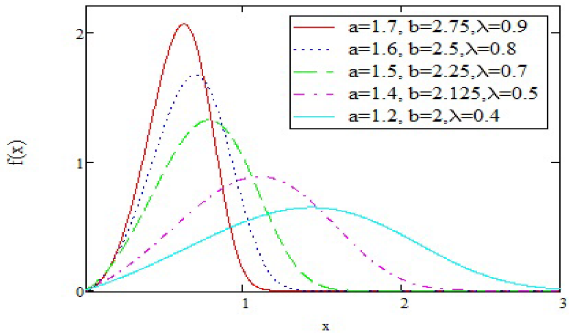

Figure 2: The pdf of the WGED.

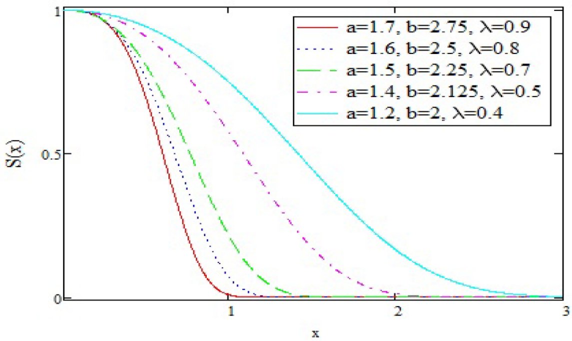

Figure 3: The survival function of the WGED.

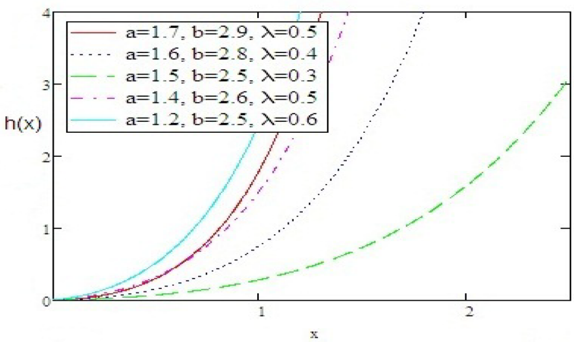

Figure 4: The hazard rate function of the WGED.

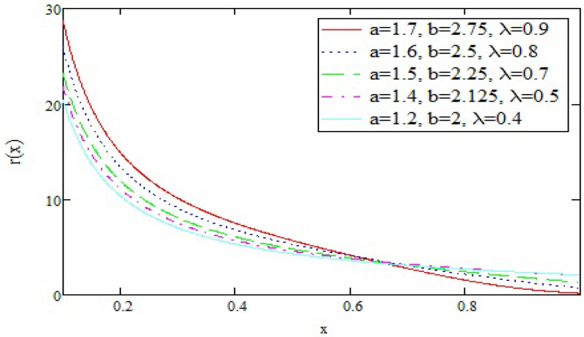

Figure 5: Reversed hazard rate function of the WGED.

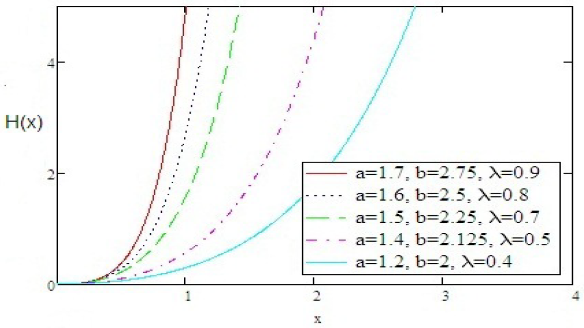

Figure 6: Cumulative hazard rate function of the WGED.

It is clear that the pdf and the hazard function have many different shapes, which allows this distribution to fit different types of lifetime data.

3 Statistical Properties

In this Section, we study the statistical properties for the WGED, specially quantile function and simulation median, skewness, kurtosis and moments.

3.1 Quantile and median

In this subsection, we determine the explicit formulas of the quantile and the median of WGED. The quantile x_q of the WGED is given by

$$F(x_q) = q, \quad 0 < q < 1. \tag{3.1}$$

From Eq. (2.1), x_q can be obtained as follows.

$$x_q = \frac{1}{\lambda} \ln \left[1 + \left(\frac{1}{a} \ln(1-q) \right)^{\frac{1}{b}} \right]. \tag{3.2}$$

Setting $q = 0.5$ in Eq. (3.2), we get the median of WGED as follows.

$$x_q = \frac{1}{\lambda} \ln \left[1 + \left(-\frac{\ln(2)}{a} \right)^{\frac{1}{b}} \right]. \tag{3.3}$$

3.2 The mode

In this subsection, we will derive the mode of the WGED by derivation its pdf with respect to x and equate it to zero. The mode is the solution the following equation with respect to x.

$$f^{'}(x) = 0. \tag{3.4}$$

By substitution pdf from Eq. (2.2) in Eq.(3.4), we have

$$\frac{d}{dx}\left[ab\lambda \cdot e^{\lambda x_i} \cdot [e^{\lambda x_i} - 1]^{b-1} \cdot e^{-a[e^{\lambda x_i}-1]^b}\right] = 0,$$
$$\frac{d}{dx}\left[h(x;a,b,\lambda) \cdot S(x;a,b,\lambda)\right] = 0,$$
$$\left[h^{'}(x;a,b,\lambda) \cdot S(x;a,b,\lambda) + h(x;a,b,\lambda) \cdot S^{'}(x;a,b,\lambda)\right] = 0,$$

then

$$\left[h^{'}(x;a,b,\lambda) - (h(x;a,b,\lambda))^2\right] S(x;a,b,\lambda) = 0, \tag{3.5}$$

where $h(x;a,b,\lambda),\ S(x;a,b,\lambda)$ are hazard function and survival function of WGED respectively.
It is not possible to get an analytic solution in x to Eq. (3.5) in the general case. It has to be obtained numerically by using methods such as fixed-point or bisection method.

3.3 Skewness and kurtosis

The analysis of the variability Skewness and Kurtosis on the shape parameters b, λ can be investigated based on quantile measures. The short comings of the classical Kurtosis measure are well-known. The Bowely's skewness [12] based on quartiles is given by

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}}, \tag{3.6}$$

and the Moors' Kurtosis [16] is based on octiles

$$K_u = \frac{q_{(0.875)} - q_{(0.625)} - q_{(0.375)} + q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}}, \tag{3.7}$$

where $q_{(.)}$ represents quantile function.

3.4 Moments

In this subsection, we discuss the rth moment for WGED. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

Theorem 3.1. *If X has WGED (a, b, λ), then the rth moments of random variable X, is given by the following*

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k} a^{i+1} \Gamma(b(i+1)+j+1)\Gamma(r+1)}{i!j!\lambda^r (k+1)^{r+1}\Gamma((b+i)+1)} \binom{b(i+1)+j+1}{k}. \tag{3.8}$$

Proof. We start with the well known distribution of the rth moment of the random variable X with pdf $f(x)$ given by

$$\mu_r^{'} = \int_0^{\infty} x^r f(x;a,b,\lambda)dx. \tag{3.9}$$

Substituting from Eq. (1.1) and Eq. (1.2) into Eq. (1.13) we get

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i a^{i+1} b \cdot \Gamma(b(i+1)+j+1)}{i!j!\Gamma((b+i)+1)} \int_0^{\infty} x^r \lambda e^{-\lambda x} \left[1-e^{-\lambda x}\right]^{b(i+1)+j-1} dx,$$

since $0 < 1 - e^{-\lambda x} < 1$ for $x > 0$, the binomial series expansion of

$$\left[1-e^{-\lambda x}\right]^{b(i+1)+j-1}$$

yields

$$\left[1-e^{-\lambda x}\right]^{b(i+1)+j-1} = \sum_{k=0}^{\infty}(-1)^k \cdot \binom{b(i+1)+j-1}{k} e^{-k\lambda x},$$

then we get

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k} a^{i+1} \lambda \Gamma(b(i+1)+j+1)}{i!j!\Gamma((b+i)+1)} \binom{b(i+1)+j-1}{k} \int_0^{\infty} x^r e^{-(k+1)\lambda x} dx,$$

by using the definition of gamma function in the form, Zwillinger [24],

$$\Gamma(Z) = x^z \int_0^{\infty} e^{tx} t^{z-1} dt, \quad z, x, > 0.$$

Finally, we obtain the rth moment of WGED in the form

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k} a^{i+1} \Gamma(b(i+1)+j+1)\Gamma(r+1)}{i!j!\lambda^r (k+1)^{r+1}\Gamma((b+i)+1)} \binom{b(i+1)+j+1}{k}.$$

This completes the proof. □

4 The Moment Generating Function

The moment generating function (mgf), $M_X(t)$, of a random variable X provides the basis of an alternative route to analytic results compared with working directly with the pdf and cdf of X.

Theorem 4.1. *The moment generating function (mgf) of WGED is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k} a^{i+1} t^r \Gamma(b(i+1)+j+1)\Gamma(r+1)}{i!j!r!\lambda^r (k+1)^{r+1}\Gamma((b+i)+1)} \binom{b(i+1)+j+1}{k}. \quad (4.1)$$

Proof. The moment generating function, $M_X(t)$, of the random variable X with pdf, $f(x)$ is given by

$$M_X(t) = \int_0^{\infty} e^{tx} f(x; a, b, \lambda) dx,$$

using series expansion of e^{xt}, we obtain

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x; a, b, \lambda) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'}. \tag{4.2}$$

Substituting from Eq. 3.8 into Eq. 4.2, we obtain the moment generating function of WGED in the form

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+k} a^{i+1} t^r \Gamma(b(i+1)+j+1)\Gamma(r+1)}{i!j!r!\lambda^r (k+1)^{r+1} \Gamma((b+i)+1)} \binom{b(i+1)+j+1}{k}.$$

This completes the proof. □

5 Order Statistics

In this Section, we derive closed form expressions for the pdf of the rth order statistic of the WGED. Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, \cdots, X_n$ which taken from a continuous population with cdf, $F(x; \varphi)$ and pdf, $f(x; \varphi)$, then the pdf of $X_{r:n}$ is given

$$f_{r:n}(x; \varphi) = \frac{1}{B(r, n-r+1)} [F(x; \varphi)]^{r-1} [1 - F(x; \varphi)]^{n-r} f(x; \varphi), \tag{5.1}$$

where $f(x; \varphi)$, $F(x; \varphi)$ are the pdf and cdf of $WGED(\varphi)$ given by Eq. (2.2) and Eq.(2.1) respectively, $\varphi = (a, b, \lambda)$ and $B(.,.)$ is the beta function, also we define first order statistics $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$, and the last order statistics as $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$. Since $0 < F(x; \varphi) < 1$ for $x > 0$, we can use the binomial expansion of $[1 - F(x; \varphi)]^{n-r}$ given as follows

$$[1 - F(x; \varphi)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \varphi)]^i. \tag{5.2}$$

Substituting from Eq. (5.2) into Eq. (5.1), we obtain

$$f_{r:n}(x; \varphi) = \frac{f(x; \varphi)}{B(r, n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \varphi)]^{i+r-1}. \tag{5.3}$$

Substituting from Eq. (2.1) and Eq. (2.2) into Eq. (5.3), we obtain

$$f_{r:n}(x; a, b, \lambda) = \sum_{i=0}^{n-r} \sum_{j=0}^{i+r-1} \frac{(-1)^{i+j} n!}{i!(r-1)!(n-r-1)!} \binom{i+r-1}{j} f(x; (j+1)a, b, \lambda). \tag{5.4}$$

Relation (5.4) show that $f_{r:n}(x; \varphi)$ is the weighted average of the WGED with different shape parameters.

6 Parameters Estimation

In this Section, point and interval estimation of the unknown parameters of the WGED are derived by using the method of maximum likelihood based on a complete sample data.

6.1 Maximum likelihood estimation:

Let $x_1, x_2, \cdots, x_n$ denote a random sample of complete data from the WGED. The likelihood function is given as

$$L = \prod_{i=1}^{n} f(x_i, a, b, \lambda), \tag{6.1}$$

substituting from (2.2) into (6.1), we have

$$L = \prod_{i=1}^{n} ab\lambda\, e^{\lambda x_i}[e^{\lambda x_i} - 1]^{b-1} e^{-a[e^{\lambda x_i}-1]^b}.$$

The log-likelihood function is

$$\mathcal{L} = n\ln(ab\lambda) + \lambda\sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n}\ln(e^{\lambda x_i} - 1) - a\sum_{i=1}^{n}[e^{\lambda x_i} - 1]^b. \tag{6.2}$$

The maximum likelihood estimation (MLE) of the parameters are obtained by differentiating the log-likelihood function $\mathcal{L}$ with respect to the parameters a, b and λ and setting the result equal to zero, we have the following normal equations.

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\left[e^{\lambda x_i} - 1\right]^b = 0, \tag{6.3}$$

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln\left(e^{\lambda x_i} - 1\right) - a\sum_{i=1}^{n}\left[e^{\lambda x_i} - 1\right]^b \ln\left(e^{\lambda x_i} - 1\right) = 0, \tag{6.4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} - 1} - ab\sum_{i=1}^{n} x_i e^{\lambda x_i}\left[e^{\lambda x_i} - 1\right]^{b-1} = 0. \tag{6.5}$$

The MLEs can be obtained by solving the nonlinear equations previous, (6.3)–(6.5), numerically for a, b and λ.

6.2 Asymptotic confidence bounds

In this Section, we derive the asymptotic confidence intervals of these parameters when $a, b > 0$ and $\lambda > 0$ as the MLEs of the unknown parameters $a, b > 0$ and $\lambda > 0$ can not be obtained in closed forms, by using variance covariance matrix I^{-1} see Lawless [13], where I^{-1} is the inverse of the observed information matrix which defined as follows

$$\begin{aligned}
\mathbf{I^{-1}} &= \begin{pmatrix} -\frac{\partial^2 \mathcal{L}}{\partial a^2} & -\frac{\partial^2 \mathcal{L}}{\partial a \partial b} & -\frac{\partial^2 \mathcal{L}}{\partial a \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial b \partial a} & -\frac{\partial^2 \mathcal{L}}{\partial b^2} & -\frac{\partial^2 \mathcal{L}}{\partial b \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial a} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial b} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} var(\hat{a}) & cov(\hat{a}, \hat{b}) & cov(\hat{a}, \hat{\lambda}) \\ cov(\hat{b}, \hat{a}) & var(\hat{b}) & cov(\hat{b}, \hat{\lambda}) \\ cov(\hat{\lambda}, \hat{a}) & cov(\hat{\lambda}, \hat{b}) & var(\hat{\lambda}) \end{pmatrix}.
\end{aligned} \tag{6.6}$$

The second partial derivatives included in I are given as follows.

$$\frac{\partial^2 \mathcal{L}}{\partial a^2} = -\frac{n}{a^2}, \tag{6.7}$$

$$\frac{\partial^2 \mathcal{L}}{\partial a \partial b} = -\sum_{i=1}^{n} \left[e^{\lambda x_i} - 1\right]^b \ln\left(e^{\lambda x_i} - 1\right), \tag{6.8}$$

$$\frac{\partial^2 \mathcal{L}}{\partial a \partial \lambda} = -b \sum_{i=1}^{n} x_i e^{\lambda x_i} \left[e^{\lambda x_i} - 1\right]^{b-1}, \tag{6.9}$$

$$\frac{\partial^2 \mathcal{L}}{\partial b^2} = -\frac{n}{b^2} - a \sum_{i=1}^{n} \left[e^{\lambda x_i} - 1\right]^b \left[\ln\left(e^{\lambda x_i} - 1\right)\right]^2, \tag{6.10}$$

$$\frac{\partial^2 \mathcal{L}}{\partial b \partial \lambda} = \sum_{i=1}^{n} \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} - 1} - a \sum_{i=1}^{n} x_i e^{\lambda x_i} [e^{\lambda x_i} - 1]^{b-1} \left[b \ln(e^{\lambda x_i} - 1) + 1\right], \tag{6.11}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = -\frac{n}{\lambda^2} - (b-1) \sum_{i=1}^{n} \frac{x_i^2 e^{\lambda x_i}}{(e^{\lambda x_i} - 1)^2} - ab \sum_{i=1}^{n} x_i^2 e^{\lambda x_i} \left(b e^{\lambda x_i} - 1\right) \left[e^{\lambda x_i} - 1\right]^{b-2}. \tag{6.12}$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters a, b and λ, by using variance matrix as in the following forms

$$\hat{a} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{a})}, \quad \hat{b} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{b})}, \quad \hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\lambda})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$-th percentile of the standard normal distribution.

7 Application

In this Section, we present the analysis of a real data set using the WGED (a, b, λ) model and compare it with the other fitted models such as ED, generalized exponential distribution (GED), [3], beta exponential distribution (BED), [17] and the beta generalized exponential distribution (BGED), [21] using Kolmogorov-Smirnov (K-S) statistic, as well as Akaike information criterion (AIC), Akaike , [1], Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) values, Schwarz [19].
The data set is obtained from Smith and Naylor [20]. The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. This data set is in Table 1.

Table 1: The data are the strengths of 1.5 cm glass fibres, [20].

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64
1.68	1.73	1.81	2	0.74	1.04	1.27	1.39	1.49
1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01	0.77
1.11	1.28	1.42	1.5	1.54	1.6	1.62	1.66	1.69
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.5	1.55
1.61	1.62	1.66	1.7	1.77	1.84	0.84	1.24	1.3
1.48	1.51	1.55	1.61	1.63	1.67	1.7	1.78	1.89

Table 2 gives MLs of parameters of the WGED and sub-models and goodness of fit statistics are in Table 3.

Table 2: MLEs of parameters, Log-likelihood.

Model	MLEs of parameters	K-S	p-value
ED	$\hat{\lambda}$=0.664	0.402316	1.44529×10^{-09}
GED	$\hat{\lambda}$=2.6105, $\hat{\alpha}$=31.3032	0.213118	0.005444
BED	$\hat{a}$=17.7786, $\hat{b}$=22.7222, $\hat{\lambda}$=0.3898	0.159819	0.07220
BGED	$\hat{a}$=0.4125, $\hat{b}$=93.4655, $\hat{\lambda}$=0.92271, $\hat{\alpha}$=22.6124	0.150611	0.10470
WGED	$\hat{a}$=56.881, $\hat{b}$=4.893, $\hat{\lambda}$=0.222	0.127366	0.24259

Table 3: Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted.

Model	$\mathcal{L}$	$-2\mathcal{L}$	AIC	AICC	BIC	HQIC
ED	-88.8300	-177.6600	179.6600	179.7256	181.8031	180.5029
GED	-31.3834	-62.7668	66.7668	66.9668	71.0531	68.4526
BED	-24.1270	-48.2540	54.2540	54.6608	60.6834	56.7827
BGED	-15.5995	-31.1990	39.1990	39.8887	47.7715	42.5706
WGED	-14.828	-29.6560	35.6560	36.0628	42.0854	38.1847

We find that the WGED with the three-number of parameters provides a better fit than the previous new modified exponential distribution which was the best in [3, 4, 17, 21]. It has the largest likelihood, and the smallest K-S, AIC, BIC and HQIC values among those considered in this paper.
Substituting the MLE's of the unknown parameters a, b, λ into (6.6), we get estimation of the variance covariance matrix as the following

$$I_0^{-1} = \begin{pmatrix} 3.655 \times 10^3 & 7.228 & -2.205 \\ 7.228 & 0.213 & 1.141 \times 10^{-3} \\ -2.205 & 1.141 \times 10^{-3} & 1.505 \times 10^{-3} \end{pmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters a, b and λ are $[0, 175.11]$, $[3.989, 57.785]$ and $[0.146, 0.298]$, respectively.
To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of a, b and λ in Figures 7 and 8.

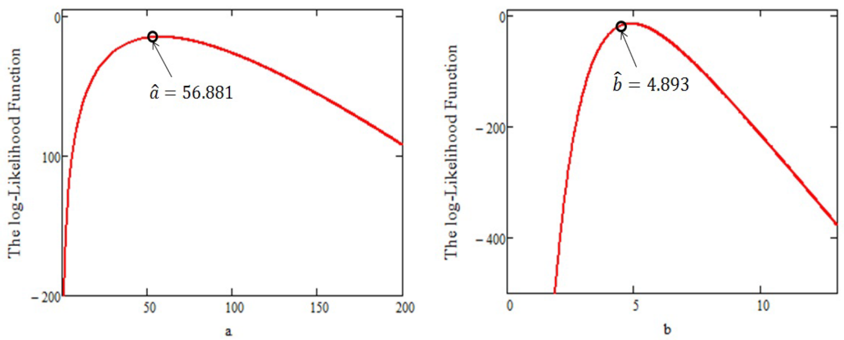

Figure 7: The profile of the log-likelihood function of a, b.

https://doi.org/10.54582/TSJ.2.2.19

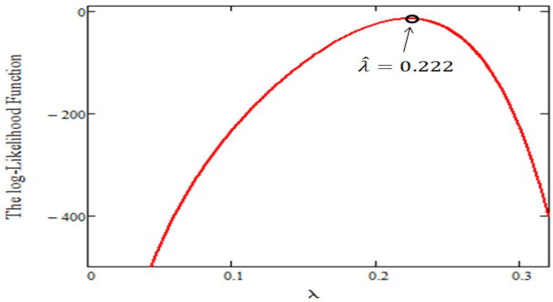

Figure 8: The profile of the log-likelihood function of λ.

The nonparametric estimate of the survival function using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be ED, GE, BED, BGED and WGED are computed and plotted in Figure 9.

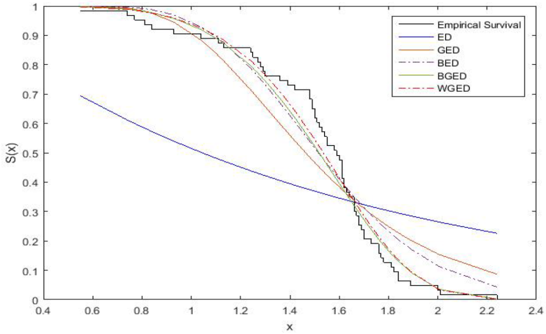

Figure 9: The Kaplan-Meier estimate of the survival function for the data.

Figure 10 gives the form of the cdf for the ED, GE, BED, BGED and WGED which are used to fit the data after replacing the unknown parameters included in each distribution by their MLEs.

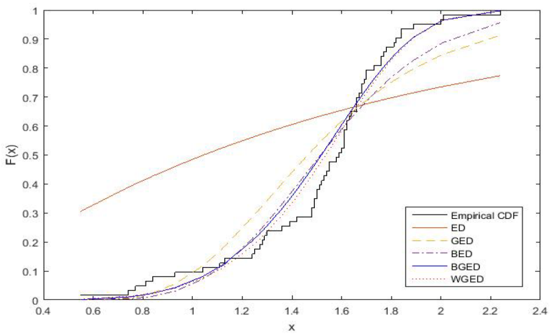

Figure 10: The Fitted cdf for the data.

8 Conclusions

A new distribution, based on Weibull- Generalized family distributions, has been proposed and its properties studied. The idea is to add parameter to ED, so that the hazard function is either increasing or more importantly, bathtub shaped. Using Weibull generator component, the distribution has flexibility to model the second peak in a distribution. We have shown that the WGED fits certain well-known data sets better than existing modifications of the generalized families of ED.

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