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Entropy solutions for some nonlinear elliptic unilateral  
problems in anisotropic Sobolev spaces

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### Abstract

In this paper, we consider the following nonlinear elliptic unilateral equations of the type

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + g(x, u, \nabla u) + |u|^{p_0-2}u = f \text{ in } \Omega$$

In the anisotropic Sobolev space, we prove the existence of entropy solutions for our unilateral problem, the function  $g(x, u, \nabla u)$  is a nonlinear lower order term with natural growth with respect to  $|\nabla u|$ , satisfying the sign condition and the datum  $f$  belongs to  $L^1(\Omega)$ .

**key words** : Anisotropic Sobolev spaces, nonlinear elliptic unilateral problem, entropy solutions.

### 1- Introduction

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$ ,  $N \geq 2$ . Boccardo and Gallouët have considered in [14] the elliptic problem

$$\begin{cases} -\text{div}a(x, u, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where  $f$  is a bounded Radon measure on  $\Omega$ . They have proved the existence of solutions  $u \in W_0^{1,q}(\Omega)$  for all  $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$ . Also, they have proved some regularity results. In [11], Boccardo has studied the existence of entropy solutions for the quasilinear elliptic problem of the type

$$\begin{cases} -\text{div}a(x, u, \nabla u) = f - \text{div}\phi(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

with  $f \in L^1(\Omega)$  and  $\phi(u) \in C^0(\mathbb{R}, \mathbb{R}^N)$ . He has proved the existence and some regularity of solutions. The most studied problems were involved in the p-Laplace operators

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u),$$

with  $p$  constant, where the authors have proved the existence of solutions for some nonlinear elliptic unilateral problems in Sobolev spaces. (see for example [13, 14, 20]).

In the framework of Orlicz Sobolev spaces, Aharouch and Bennouna [1] have treated the quasilinear elliptic unilateral problem

$$\begin{cases} -\text{div} a(x, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

where  $f \in L^1(\Omega)$ . They have proved the existence and uniqueness of entropy solutions  $u \in W_0^1 L_M(\Omega)$  without any restriction on the  $N$ -function  $M$  of the Orlicz spaces (i.e. without assuming  $\Delta_2$ -condition), we refer also to [5, 10, 16] for more details.





Recently, the anisotropic Sobolev spaces have attracted the attention of many scientists and researchers (see. [7, 18, 22]), this impulse mainly comes from their physical applications in the processes of image restoration, ows of electro-rheological fluids and thermistor problem.

M. AL-hawmi, E.Azroul, H. Hjjaj and A.Touzani have studied in [2] the existence of entropy solutions for some anisotropic quasilinear elliptic unilateral problems

$$\begin{cases} \mathbf{A}u = \mu - \operatorname{div}\phi(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.4)$$

where  $\mathbf{A}u$  is an operator of Leray-Lions type acting from  $W_0^{1,\vec{p}}(\Omega)$  into its dual define by

$$\mathbf{A}u = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$

where  $\phi(u) \in C^0(\mathbb{R}, \mathbb{R}^N)$  and  $\mu = f - \operatorname{div} F$  such that  $f \in L^1(\Omega)$  and  $F \in \prod_{i=1}^N L^{p_i}(\Omega)$ .

L. M. Kozhevnikova has proved that the entropy solution obtained is a renormalized solution in [19] of the problem under consideration.

$$\begin{cases} \sum_{i=1}^N (a_i(x, u, \nabla u))_{x_i} = |u|^{p(x)-2}u + b(x, u, \nabla u) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

AL-hawmi, A. Benkirane, H. Hjjaj and A. Touzani have studied in [3] the existence and uniqueness of entropy solutions for some nonlinear elliptic unilateral problems

$$\begin{cases} \mathbf{A}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.6)$$

in Musielak-Orlicz-Sobolev spaces, where  $f \in L^1(\Omega)$  and

$A : D(A) \subset W_0^1 L_\phi(\Omega) \mapsto W_0^1 L_\psi(\Omega)$  is the Leray-Lions operator defined as:

$$\mathbf{A}u = -\operatorname{div} a(x, \nabla u)$$

In this paper, we will prove the existence of entropy solutions for nonlinear anisotropic unilateral elliptic problem of the type

$$\begin{cases} \mathbf{A}u + g(x, u, \nabla u) + |u|^{p_0-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.7)$$

where  $\mathbf{A}u$  is an operator of Leray-Lions type acting from  $W_0^{1,\vec{p}}(\Omega)$  into its dual define by

$$\mathbf{A}u = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$





and the function  $g(x, u, \nabla u)$  is a non linear lower order term with natural growth with respect to  $|\nabla u|$  satisfying the sign condition, that is  $g(x, u, \nabla u) u \geq 0$ .

This paper is organized as follows. In section 2 we recall some definitions and basic properties concerning the anisotropic Sobolev spaces. We introduce in section 3 the assumptions on  $a_i(x, u, \nabla u)$  and  $g(x, u, \nabla u)$  for which our problem has at least one solution. The section 4 will be devoted to show the existence of entropy solutions for our anisotropic nonlinear elliptic unilateral problem (1.7).

## 2- Preliminary

Let  $\Omega$  be an open bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$  with boundary  $\partial\Omega$ . Let  $p_0, p_1, \dots, p_N$  be  $N + 1$  exponents, with  $1 < p_i < \infty$  for  $i = 1, \dots, N$ . We denote

$$\bar{p} = (p_0, p_1, \dots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \dots, N$$

We set

$$\underline{p} = \min \{p_0, p_1, \dots, p_N\} \quad \text{then} \quad \underline{p} > 1. \quad (2.1)$$

The anisotropic Sobolev space  $W^{1, \bar{p}}(\Omega)$  is defined as follows

$W^{1, \bar{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \text{ and } D^i u \in L^{p_i}(\Omega) \text{ for } i = 1, 2, \dots, N\}$   
endowed with the norm

$$\|u\|_{1, \bar{p}} = \sum_{i=1}^N \|D^i u\|_{L^{p_i}(\Omega)}. \quad (2.2)$$

The space  $(W^{1, \bar{p}}(\Omega), \|u\|_{1, \bar{p}})$  is a separable and reflexive Banach space (cf [22]).

We define also  $W_0^{1, \bar{p}}(\Omega)$  as the closure of  $C_0^\infty(\Omega)$  in  $W^{1, \bar{p}}(\Omega)$  with respect to the norm (2.2).

Now, we will present some important compacts embedding,

**Lemma 2.1** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$ , then the following embedding are compact

- if  $\underline{p} < N$  then  $W_0^{1, \bar{p}}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \underline{p}^*[,$   
where  $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N},$
- if  $\underline{p} = N$  then  $W_0^{1, \bar{p}}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, \infty[ ,$
- if  $\underline{p} > N$  then  $W_0^{1, \bar{p}}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^0(\bar{\Omega}).$

The proof of this lemma follows from the fact that the embedding  $W_0^{1, \bar{p}}(\Omega) \hookrightarrow W_0^{1, \underline{p}}(\Omega)$  is continuous, and in view of the compact embedding theorem for Sobolev spaces.

**Definition 2.1** Let  $k > 0$ , we consider the truncation function  $T_k(\cdot): \mathbb{R} \rightarrow \mathbb{R}$  given by





$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| \geq k, \end{cases}$$

and we define

$$T_0^{1,\vec{p}}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \text{ measurable, } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}$$

**Proposition 2.1**  $u \in T_0^{1,\vec{p}}(\Omega)$ , for  $i = 1, \dots, N$ , exists a unique measurable function  $v_i : \Omega \rightarrow \mathbb{R}$  such that

$$D^i T_k(u) = v_i x_{\{|u| < k\}} \text{ a.e. in } \Omega, \text{ for any } k > 0,$$

where  $x_A$  denotes the characteristic function of a measurable set  $A$ . The functions  $v_i$  are called the weak partial derivatives of  $u$  and are still denoted  $D^i u$ . Moreover, if  $u$  belongs to  $W_0^{1,1}(\Omega)$ , then  $v_i$  coincides with the standard distributional derivatives of  $u$ , that is  $v_i = D^i u$ .

The proof follows the usual techniques developed in [12] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8] and [15].

### 3 - Essential assumptions

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$  ( $N \geq 2$ ), we assume that the vector  $\vec{p} = (p_0, p_1, \dots, p_N)$  satisfying the requirements that  $1 < p_i < \infty$  for  $i = 1, 2, \dots, N$ .

Taking  $\psi$  as a measurable function on  $\Omega$  with values in  $\bar{\mathbb{R}}$  such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

We define the convex subset  $K_\psi$  by

$$K_\psi = \{v \in W_0^{1,\vec{p}}(\Omega) \text{ such that } v \geq \psi \text{ a.e. in } \Omega\}.$$

We consider a Leray-Lions operator  $A : W_0^{1,\vec{p}}(\Omega) \rightarrow W^{-1,\vec{p}}(\Omega)$  given by

$$Au = - \sum_{i=1}^N D^i a_i(x, u, \nabla u)$$

where  $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  are Carathéodory functions, for  $i = 1, 2, \dots, N$ , (measurable with respect to  $x$  in  $\Omega$  for every  $(s, \xi)$  in  $\mathbb{R} \times \mathbb{R}^N$  and continuous with respect to  $(s, \xi)$  in  $\mathbb{R} \times \mathbb{R}^N$  for almost every  $x$  in  $\Omega$ ), which satisfy the following conditions

$$|a_i(x, s, \xi)| \leq \beta(R_i(x) + |s|^{p_i-1} + |\xi|^{p_i-1}) \text{ for } i = 1, 2, \dots, N, \quad (3.1)$$

$$a_i(x, s, \xi) \xi_i \geq \alpha |\xi_i|^{p_i} \text{ for } i = 1, 2, \dots, N, \quad (3.2)$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0 \text{ for } \xi_i \neq \xi'_i \quad (3.3)$$

for a.e.  $x$  in  $\Omega$  and all  $(s, \xi)$  in  $\mathbb{R} \times \mathbb{R}^N$ , where  $R_i(x)$  is a nonnegative function lying in  $L^{p'_i}(\Omega)$  and  $\alpha, \beta > 0$ .

The nonlinear term  $g(x, s, \xi)$  is a Carathéodory function which satisfies

$$g(x, s, \xi) s \geq 0 \quad (3.4)$$

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + \sum_{i=1}^N |\xi_i|^{p_i}) \quad (3.5)$$

where  $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous nondecreasing function,  $C \in L^1(\Omega)$  We consider the anisotropic nonlinear elliptic unilateral problem





$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{p_0-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3.6)$$

Where

$$f \in L^1(\Omega) \quad (3.7)$$

Now, we recall some important Lemmas useful to prove our main result.

**Lemma 3.1** (see [17], Theorem 13.47 page 216) Let  $(u_n)_n$  be a sequence in  $L^1(\Omega)$  and  $u \in L^1(\Omega)$  such that

- (i)  $u_n \rightarrow u$  a.e. in  $\Omega$ ,
- (ii)  $u_n \geq 0$  and  $u \geq 0$  a.e. in  $\Omega$ ,
- (iii)  $\int_{\Omega} u_n dx \rightarrow \int_{\Omega} u dx$ ,

then  $u_n \rightarrow u$  in  $L^1(\Omega)$ .

**Lemma 3.2** (see. [7] page 6) Assuming that (3.1) - (3.3) hold, and let  $(u_n)_{n \in \mathbb{N}}$  be a sequence in  $W_0^{1,\vec{p}}(\Omega)$  such that  $u_n \rightarrow u$  in  $W_0^{1,\vec{p}}(\Omega)$  and

$$\begin{aligned} \int_{\Omega} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u) dx \\ + \sum_{i=1}^N \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D^i u_n - D^i u) dx \rightarrow 0, \end{aligned} \quad (3.8)$$

then  $u_n \rightarrow u$  in  $W_0^{1,\vec{p}}(\Omega)$  for a subsequence.

#### 4- Main results.

We first give the definition of entropy solutions for our problem (3,6) as follows.

**Definition 4.1** A measurable function  $u$  is called entropy solution of the anisotropic nonlinear elliptic unilateral problem (3.6) if  $T_k(u) \in K_{\psi}$  and satisfy

$$\left\{ \begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) D^i T_k(u_n - v) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u_n - v) dx \\ \int_{\Omega} |u|^{p_0-2} u T_k(u_n - v) dx \leq \int_{\Omega} f T_k(u_n - v) dx \\ \text{for any } v \in W_0^{1,\vec{p}}(\Omega) \cap L^{\infty}(\Omega) \end{aligned} \right. \quad (4.1)$$

Our objective is to prove the following existence theorem.

**Theorem 4.1** Assuming that (3.1) - (3.5) and (3.7) hold, then the problem (3,6) has at least one entropy solution.

#### Proof of Theorem 4.1

##### Step 1: Approximate problems.





Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $W^{-1, \bar{p}'}(\Omega) \cap L^1(\Omega)$  such that  $f_n \rightarrow f$  in  $L^1(\Omega)$   
and

$|f_n| \leq |f|$  (for example  $f_n = T_n(f)$ ). We consider the approximate problem.

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i(u_n - v) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n)(u_n - v) dx$$

$$+ \int_{\Omega} |u_n|^{p_0-2} u_n (u_n - v) dx \leq \int_{\Omega} f_n (u_n - v) dx \quad (4.2)$$

for any  $v \in W_0^{1, \bar{p}}(\Omega) \cap L^{\infty}(\Omega)$

for any  $v \in K_{\psi}$ , and  $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}$ ,  $g_n(x, s, \xi) s \geq 0$ ,

$|g_n(x, s, \xi)| \leq |g(x, s, \xi)|$  and  $|g_n(x, s, \xi)| \leq n$ . We define the operators  $A_n$   
and  $G_n$  acted from  $W_0^{1, \bar{p}}(\Omega)$  into its dual  $W^{-1, \bar{p}'}(\Omega)$  by

$$\langle A_n u, v \rangle = \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i v dx + \int_{\Omega} |u|^{p_0-2} u v dx$$

$$\forall u, v \in W_0^{1, \bar{p}}(\Omega)$$

and

$$\langle G_n u, v \rangle = \int_{\Omega} g(x, u, \nabla u) v dx \quad \forall u, v \in W_0^{1, \bar{p}}(\Omega).$$

**Lemma 4.1** The operator  $B_n = A_n + G_n$  from  $W_0^{1, \bar{p}}(\Omega)$  into  $W^{-1, \bar{p}'}(\Omega)$  is  
pseudo-monotone. Moreover,  $B_n$  is coercive in the following sense: there  
exists  $v_0 \in K_{\psi}$  such that

$$\frac{\langle B_n u, v - v_0 \rangle}{\|u\|_{1, \bar{p}}} \rightarrow \infty \quad \text{as } \|u\|_{1, \bar{p}} \rightarrow \infty \quad \text{for } v \in K_{\psi}.$$

In view of **Lemma 5.1** (see **Appendix**), there exists at least one solution  
 $u_n \in W_0^{1, \bar{p}}(\Omega)$  of the anisotropic nonlinear elliptic unilateral problem (4.2) (cf.  
[21], **Theorem 8.2 page131**).

### Step 2: A priori estimates.

Let  $k \geq \max(1, \|\psi^+\|_{\infty})$ ; we set  $v = u_n - \eta T_k(u_n - \psi^+) W_0^{1, \bar{p}}(\Omega) \cap$   
 $L^{\infty}(\Omega)$  for  $\eta$  small enough we have  $v \in K_{\psi}$ , thus  $v$  is an admissible test  
function in (4.2), and we obtain

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \psi^+) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - \psi^+) dx$$

$$+ \int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \leq \int_{\Omega} f_n T_k(u_n - \psi^+) dx \quad (4.3)$$





From (4.3), it follows that

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx + \int_{\{|u_n - \psi^+| \leq k\}} g_n(x, u_n, \nabla u_n) u_n dx \\ & \quad + \int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \leq \int_{\Omega} f_n T_k(u_n - \psi^+) dx \quad (4.4) \\ & + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i \psi^+ dx + \int_{\{|u_n - \psi^+| \leq k\}} g_n(x, u_n, \nabla u_n) \psi^+ dx \end{aligned}$$

Since  $k \geq \|\psi^+\|_{\infty}$  then  $T_k(u_n - \psi^+)$  have the same sign as  $u_n$  on the set  $\{|u_n - \psi^+| > k\}$  it follows that

$$\begin{aligned} & \int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \\ & = \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-2} u_n (u_n - v) dx \\ & \quad + \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) dx \\ & \geq \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0} - \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| dx \end{aligned}$$

In view of (3,2), we obtain

$$\begin{aligned} & \propto \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} D^i u_n + \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0} \\ & \leq \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx \\ & \quad + \int_{\{|u_n - \psi^+| \leq k\}} g_n(x, u_n, \nabla u_n) u_n dx \\ & \leq \int_{\Omega} f_n T_k(u_n - \psi^+) dx \quad (4.5) \\ & \quad + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)| |D^i \psi^+| dx \\ & \quad + \int_{\{|u_n - \psi^+| \leq k\}} |g_n(x, u_n, \nabla u_n)| |\psi^+| dx \\ & \quad + \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| dx \end{aligned}$$

Thanks to Young's inequality we have







$$\int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| dx \leq \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} |\psi^+| dx + C_1 \quad (4.6)$$

also, in view of (3,1) and Young's inequality, we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)| |D^i \psi^+| dx \\ & \leq \beta \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} R_i(x) |D^i \psi^+| dx \\ & + \beta \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p-1} |D^i \psi^+| dx \\ & + \beta \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p^{i-1}} |D^i \psi^+| dx \leq C_2 \quad (4.7) \\ & + C_3 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^p dx + \frac{\alpha}{4} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p^i} dx \\ & + C_4 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i \psi^+|^p dx + C_5 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i \psi^+|^{p^i} dx, \end{aligned}$$

also, in view of (3.5) and Young's inequality, we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |g_n(x, u_n, \nabla u_n)| |D^i \psi^+| dx \\ & \leq \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} b|u_n|(c(x) + |D^i u_n|^{p^{i-1}}) |D^i \psi^+| dx \\ & \leq C_6 + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p^{i-1}} |D^i \psi^+| dx \quad (4.8) \\ & \leq C_6 + \frac{\alpha}{2} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p^i} dx \\ & + C_7 \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i \psi^+|^{p^i} dx, \end{aligned}$$

By combining (4.5) - (4.8), we deduce that

$$\begin{aligned} & \frac{\alpha}{4} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p^i} dx + \frac{1}{2} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} dx \quad (4.9) \\ & \leq k \|f\|_1 + N \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^p dx + C_8 \end{aligned}$$





Then, there exists a constant  $C_9$  depending only on  $k$  such that

$$\frac{\alpha}{4} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |D^i u_n|^{p_i} dx \leq C_9(k) \quad (4.10)$$

and since

$$\{x \in \Omega, |u_n| \leq k\} \subset \{x \in \Omega, |u_n - \psi^+| \leq k + \|\psi^+\|_\infty\}$$

Therefore

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i} dx &= \sum_{i=1}^N \int_{\{|u_n| \leq k\}} |D^i u_n|^{p_i} dx + \int_{\Omega} |T_k(u_n)|^{p_0} dx \\ &\leq \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k + \|\psi^+\|_\infty\}} |D^i u_n|^{p_i} dx + |k|^{p_0} |\Omega| \\ &\leq C_{10}(k, \|\psi^+\|_\infty), \end{aligned}$$

and we obtain

$$\|T_k(u_n)\|_{1, \bar{p}} \leq C_{11}(k, \|\psi^+\|_\infty),$$

where  $C_{11}$  is positive constant that does not depend on  $n$ . Thus, the sequence  $(T_k(u_n))_n$  is bounded in  $W_0^{1, \bar{p}}(\Omega)$  uniformly in  $n$ , then there exists a subsequence still denoted  $(T_k(u_n))_{n \in N}$  and a function  $v_k \in W_0^{1, \bar{p}}(\Omega)$

such that

$$\left\{ \begin{array}{l} T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^{1, \bar{p}}(\Omega), \\ T_k(u_n) \rightarrow v_k \text{ strongly in } L^{\bar{p}}(\Omega) \text{ and a.e in } \Omega \end{array} \right\} \quad (4.11)$$

On the other hand, thanks to (4.9) and Poincaré inequality, we have

$$\begin{aligned} \|\nabla T_k(u_n)\|_p^p &= \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^p dx \leq \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i} dx + N|\Omega| \\ &\leq \frac{4}{\alpha} k \|f\|_1 + \frac{4}{\alpha} N \|T_k(u_n)\|_p^p + C_{12} \\ &\leq \frac{4}{\alpha} k \|f\|_1 + \frac{4}{\alpha} CN \|\nabla T_k(u_n)\|_p^p + C_{12} \end{aligned} \quad (4.12)$$

Therefore, by taking  $\frac{4}{\alpha} CN \leq \frac{1}{3}$ , there exists a constant  $C_{13}$  that does not depend on  $k$  and  $n$ , such that

$$\|\nabla T_k(u_n)\|_p \leq C_{13} k^{\frac{1}{3}} \quad \text{for } k \geq 1$$

and we obtain

$$\begin{aligned} k \text{ meas } \{|u_n| > k\} &= \int_{\{|u_n| > k\}} |T_k(u_n)| dx \leq \int_{\Omega} |T_k(u_n)| dx \\ &\leq \|1\|_{p'} \|T_k(u_n)\|_p \end{aligned} \quad (4.13)$$





$$\leq C \|\nabla T_k(\mathbf{u}_n)\|_p \leq C_{14} k^{\frac{1}{p}},$$

which yields

$$\text{meas} \{|\mathbf{u}_n| > k\} \leq C_{14} \frac{1}{k^{\frac{1}{p}}} \rightarrow \infty, \quad k \rightarrow \infty \quad (4.14)$$

Now, we will show that the sequence  $(\mathbf{u}_n)_n$  is a Cauchy sequence in measure. Indeed, we have for every  $\delta > 0$ ,

$$\begin{aligned} \text{meas} \{|\mathbf{u}_n - \mathbf{u}_m| > \delta\} \\ \leq \text{meas} \{|\mathbf{u}_n| > k\} + \text{meas} \{|\mathbf{u}_m| > k\} \\ + \text{meas} \{|T_k(\mathbf{u}_n) - T_k(\mathbf{u}_m)| > \delta\} \end{aligned}$$

Let  $\varepsilon > 0$ , in view of (4.14) we may choose  $k = k(\varepsilon)$  large enough such that

$$\text{meas} \{|\mathbf{u}_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas} \{|\mathbf{u}_m| > k\} \leq \frac{\varepsilon}{3} \quad (4.15)$$

Moreover, thanks to (4.11) we have

$$T_k(\mathbf{u}_n) \rightarrow \eta_k \text{ strongly in } L^p(\Omega) \text{ and a.e in } \Omega$$

Thus  $(T_k(\mathbf{u}_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in measure, and for any  $k > 0$  and  $\delta, \varepsilon > 0$ , there exists  $n_0 = n_0(k, \delta, \varepsilon)$  such that

$$\text{meas} \{|T_k(\mathbf{u}_n) - T_k(\mathbf{u}_m)| > \delta\} \leq \frac{\varepsilon}{3} \text{ for all } m, n \geq n_0(k, \delta, \varepsilon) \quad (4.16)$$

By combining (4.15) and (4.16), we conclude that : for all  $\delta, \varepsilon > 0$ , there exists  $n_0 = n_0(\delta, \varepsilon)$  such that

$$\text{meas} \{|\mathbf{u}_n - \mathbf{u}_m| > \delta\} \leq \varepsilon \quad \text{for any } m, n \geq n_0.$$

It follows that  $(\mathbf{u}_n)_n$  is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function  $\mathbf{u}$ . Thanks to (4.11) we obtain

$$\left\{ \begin{array}{l} T_k(\mathbf{u}_n) \rightharpoonup T_k(\mathbf{u}) \quad \text{weakly in } W_0^{1,p}(\Omega), \\ T_k(\mathbf{u}_n) \rightarrow T_k(\mathbf{u}) \quad \text{strongly in } L^p(\Omega) \text{ and a.e in } \Omega \end{array} \right\} \quad (4.17)$$

### Step 3: Convergence of the gradient.

In the sequel, we denote by  $\varepsilon_i(n)$   $i = 1, 2 \dots$  a various functions of real numbers which converges to 0 as  $n$  tends to infinity.

Let  $\varphi_k(s) = s \cdot e^{\gamma s^2}$  where  $\gamma = \left(\frac{b(k)}{2\alpha}\right)^2$ . It is obvious that

$$\varphi'_k(s) - \frac{b(k)}{\alpha} |\varphi_k(s)| \geq \frac{1}{2} \quad \forall s \in \mathbb{R}.$$

We also consider  $\mathbf{h} > \mathbf{k} > \mathbf{0}$  and  $\mathbf{M} = 4\mathbf{k} + \mathbf{h}$  and we set

$$\omega_n = T_{2k}(\mathbf{u}_n - T_h(\mathbf{u}_n)) + T_k(\mathbf{u}_n) - T_k(\mathbf{u}).$$

By taking  $\mathbf{v} = \mathbf{u}_n - \eta \varphi_k(\omega_n)$  we have  $\mathbf{v} \geq \psi$  for  $\eta$  small enough, thus  $\mathbf{v}$  is an admissible test function in (4.2), and we obtain





$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) \varphi'_k(\omega_n) D^i \omega_n dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx + \int_{\Omega} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx \leq \int_{\Omega} f_n \varphi_k(\omega_n) dx$$

It is easy to see that  $D^i \omega_n = 0$  on  $\{|u_n| > M\}$  and since

$g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n) \geq 0$  on  $\{|u_n| > k\}$ , then

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx + \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx \leq \int_{\Omega} f_n \varphi_k(\omega_n) dx \quad (4.18)$$

**First estimate :**

We have

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\ &= \sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i T_{2k}(u_n - T_k(u)) dx \\ &+ \sum_{i=1}^N \int_{\{|u_n| > k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n \end{aligned} \quad (4.19)$$

On one hand, since  $|u_n - T_k(u)| \leq 2k$  on  $\{|u_n| \leq k\}$ , then

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) D^i T_{2k}(u_n - T_k(u)) dx \\ &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx \\ &+ \sum_{i=1}^N \int_{\{|u_n| > k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) D^i T_k(u) dx \end{aligned} \quad (4.20)$$

Since  $1 \leq \varphi'_k(\omega_n) \leq \varphi'_k(2k)$ , then

$$\begin{aligned} & \left| \sum_{i=1}^N \int_{\{|u_n| > k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) D^i T_k(u) dx \right| \\ & \leq \varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n| > k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \end{aligned}$$





and since  $(|a_i(x, T_k(u_n), \nabla T_k(u_n))|)_{n \in \mathbb{N}}$  is bounded in  $L^{p'_i}(\Omega)$ , then there exists  $\vartheta \in L^{p'_i}(\Omega)$  such that  $(|a_i(x, T_k(u_n), \nabla T_k(u_n))|) \rightarrow \vartheta$  in  $L^{p'_i}(\Omega)$ . Therefore

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \\ & \rightarrow \sum_{i=1}^N \int_{\{|u_n|>k\}} \vartheta |D^i T_k(u)| dx = 0 \end{aligned}$$

It follows that

$$\sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i T_k(u) dx = \varepsilon_0(n) \quad (4.21)$$

On the other hand, for the second term on the right hand side of (4.19), taking  $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$ , then

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\ & = \sum_{i=1}^N \int_{\{|u_n|>k\} \cap \{|z_n| \leq 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i z_n dx \\ & + \sum_{i=1}^N \int_{\{|u_n|>k\} \cap \{|z_n| \leq 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i (u_n - T_k(u)) \chi_{\{|u_n|>h\}} dx \\ & - \sum_{i=1}^N \int_{\{|u_n|>k\} \cap \{|z_n| \leq 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i T_k(u) \chi_{\{|u_n| \leq h\}} dx \\ & \geq -\varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \end{aligned}$$

In the same way as for (4.21), we can prove that

$$\varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx = \varepsilon_1(n). \quad (4.22)$$

After adding (4.19) - (4.22), we obtain

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\ & \geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx + \varepsilon_1(n) \end{aligned} \quad (4.23)$$

which is equivalent to say



$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx \\
 & \leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\
 & - \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) \varphi'_k(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) dx - \varepsilon_2(n) \\
 & \leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx \\
 & + \varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx - \varepsilon_2(n)
 \end{aligned}$$

Concerning the second term on the right-hand side of (4.24), by applying the Lebesgue convergence theorem, we have  $T_k(u_n) \rightarrow T_k(u)$  in  $L^{p_0}(\Omega)$ , then,  $|a_i(x, T_k(u_n), \nabla T_k(u))| \rightarrow |a_i(x, T_k(u), \nabla T_k(u))|$  in  $L^{p_i}(\Omega)$ , and since  $D^i T_k(u_n)$  converges to  $D^i T_k(u)$  weakly in  $L^{p_i}(\Omega)$ , we obtain

$$\varepsilon_3(n) = \varphi'_k(2k) \sum_{i=1}^N \int_{\{|u_n|>k\}} |a_i(x, T_k(u_n), \nabla T_k(u_n))| |D^i T_k(u)| dx \rightarrow 0$$

as  $n \rightarrow \infty$  (4.25)

We conclude that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) \varphi'_k(\omega_n) (D^i T_k(u_n) \\
 & - D^i T_k(u)) dx \\
 & \leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx + \varepsilon_3(n)
 \end{aligned}$$

(4.26)

**Second estimate :**

Now, we treat the second term on the left-hand side of (4.18). From (3.5), we obtain

$$\begin{aligned}
 & \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right| \\
 & \leq \int_{\{|u_n| \leq k\}} b(|u_n|) \left( c(x) + \sum_{i=1}^N |D^i T_k(u_n)|^{p_i} \right) |\varphi_k(\omega_n)| dx \\
 & \leq b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx
 \end{aligned}$$





$$\begin{aligned}
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u_n) |\varphi_k(\omega_n)| dx \\
 & \leq b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx \\
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| dx \\
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
 & \quad (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & + \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 \text{therefore,} \\
 & \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
 & \quad (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & \geq \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right| - b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx \\
 & - \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\
 & - \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| dx. \tag{4.27}
 \end{aligned}$$

We have  $\varphi_k(\omega_n) \rightarrow \varphi_k(T_{2k}(u - T_h(u)))$  weak-\* in  $L^\infty(\Omega)$  as  $n \rightarrow \infty$  as, the

$$\int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)| dx \rightarrow \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(T_{2k}(u - T_h(u)))| dx = 0 \tag{4.28}$$

Concerning the third term on the right-hand side of (4.27), thanks to (4.25), we have

$$\begin{aligned}
 & \left| \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \right| \\
 & \leq \varphi_k(2k) \sum_{i=1}^N \int_{\Omega} |a_i(x, T_k(u_n), \nabla T_k(u))| |D^i T_k(u_n) - D^i T_k(u)| dx \\
 & \rightarrow 0 \text{ as } n \rightarrow \infty \tag{4.29}
 \end{aligned}$$

For the last term on the right-hand side of (4.27), we have that

$(a_i(x, T_k(u_n), \nabla T_k(u_n)))_{n \in \mathbb{N}}$  is bounded in  $L^{p_i}(\Omega)$ , then there exists  $\psi_i \in L^{p_i}(\Omega)$  such that  $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \psi_i$  in  $L^{p_i}(\Omega)$  and, using the





fact that

$$D^i T_k(u) |\varphi_k(\omega_n)| \rightarrow D^i T_k(u) \varphi_k(T_{2k}(u - T_h(u))) \quad \text{in } L^{p_i}(\Omega)$$

it follows that

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| dx \\ \rightarrow \sum_{i=1}^N \int_{\Omega} \psi_i D^i T_k(u) |\varphi_k(T_{2k}(u - T_h(u)))| dx = 0 \end{aligned} \quad (4.30)$$

By combining (4.27) - (4.30), we get

$$\begin{aligned} \frac{b(k)}{\alpha} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx \\ \geq \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right| + \varepsilon_5(n) \end{aligned} \quad (4.31)$$

### Third estimate :

We have

$$\begin{aligned} \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx \\ = \int_{\Omega} |T_k(u_n)|^{p_0-2} T_k(u_n) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ - \int_{\{|u_n| > k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ = \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) \\ - T_k(u)) e^{\gamma \omega_n^2} dx \\ + \int_{\Omega} |T_k(u)|^{p_0-2} T_k(u) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ - \int_{\{|u_n| > k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) (T_k(u_n) - T_k(u)) e^{\gamma \omega_n^2} dx \\ \geq \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) (T_k(u_n) \\ - T_k(u)) dx \\ - e^{\gamma (2k)^2} \int_{\Omega} |T_k(u)|^{p_0-1} |T_k(u_n) - T_k(u)| dx \\ - e^{\gamma (2k)^2} \int_{\{|u_n| > k\}} k^{p_0-1} |T_k(u_n) - T_k(u)| dx \end{aligned}$$

and as  $T_k(u_n) \rightarrow T_k(u)$  in  $L^{p_0}(\Omega)$ , then the second and the last term on the right-hand side of the inequality above converges to 0 as  $n$  goes to infinity.

Therefore, we get







$$\int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u))(T_k(u_n) - T_k(u)) dx$$

$$\leq \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n (T_k(u_n) - T_k(u)) \varphi_k(\omega_n) dx + \varepsilon_6(n) \quad (4.32)$$

Thanks to (4.26), (4.31) and (4.32), we obtain

$$\frac{1}{2} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)| dx$$

$$+ \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u))(T_k(u_n) - T_k(u)) dx$$

$$\leq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi'_k(\omega_n) D^i \omega_n dx$$

$$- \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) dx \right|$$

$$+ \int_{\{|u_n| \leq k\}} |u_n|^{p_0-2} u_n \varphi_k(\omega_n) dx + \varepsilon_7(n)$$

$$\leq \int_{\Omega} f_n \varphi_k(\omega_n) dx + \varepsilon_7(n) \quad (4.33)$$

We have

$$\int_{\Omega} f_n \varphi_k(\omega_n) dx = \int_{\Omega} f_n \varphi_k(u - T_h(u)) dx + \varepsilon_8(n) \quad (4.34)$$

By combining (4.33) and (4.34), we obtain

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (D^i T_k(u_n) - D^i T_k(u)) dx$$

$$+ \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u))(T_k(u_n) - T_k(u)) dx$$

$$\leq \int_{\Omega} f_n \varphi_k(u - T_h(u)) dx + \varepsilon_9(n) \quad (4.35)$$

Then, by letting  $h$  and  $n$  goes to infinity in (4.35), we get

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (D^i T_k(u_n) - D^i T_k(u)) dx$$

$$+ \int_{\Omega} (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u))(T_k(u_n) - T_k(u)) dx \rightarrow 0$$

$$(4.36)$$

Using the **Lemma 3.2**, we deduce that

$$T_k(u_n) \rightarrow T_k(u) \text{ in } W_0^{1,\vec{p}}(\Omega) \quad (4.37)$$

Therefore,

$$D^i T_k(u_n) \rightarrow D^i T_k(u) \text{ a. e. in } \Omega$$





**Step 4: The equi-integrability of  $g(x, u_n, \nabla u_n)$  and  $|u_n|^{p_0-2}u_n$ .**

In order to pass to the limit in the approximate equation, we show that

$$g(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ and } |u_n|^{p_0-2}u_n \rightarrow |u|^{p_0-2}u \text{ strongly in } L^1(\Omega)$$

By using Vitali's Theorem, it suffices to prove that  $g(x, u_n, \nabla u_n)$  and  $|u_n|^{p_0-2}u_n$  are uniformly equi-integrable. Indeed, taking  $T_1(u_n - T_h(u_n))$  as a test function in (4.2), and using (3,2) since  $T_1(u_n - T_h(u_n))$  have the same sign as  $u_n$ , we obtain

$$\begin{aligned} & \int_{\{h > |u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n)) dx \\ & + \int_{\{h > |u_n|\}} |u_n|^{p_0-2}u_n T_1(u_n - T_h(u_n)) dx \\ & \leq \int_{\{h > |u_n|\}} f_n T_1(u_n - T_h(u_n)) dx \end{aligned} \quad (4.38)$$

It follows that

$$\begin{aligned} & \int_{\{h+1 \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| dx \\ & + \int_{\{h+1 \leq |u_n|\}} |u_n|^{p_0-1} dx \\ & \leq \int_{\{h < |u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n)) dx \\ & + \int_{\{h < |u_n|\}} |u_n|^{p_0-2}u_n T_1(u_n - T_h(u_n)) dx \\ & \leq \int_{\{h < |u_n|\}} f_n T_1(u_n - T_h(u_n)) dx \leq \int_{\{h < |u_n|\}} |f| dx \end{aligned}$$

thus, for all  $\eta > 0$ , there exists  $h(\eta) > 0$  such that

$$\int_{\{h(\eta) \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{\{h(\eta) \leq |u_n|\}} |u_n|^{p_0-1} dx \leq \frac{\eta}{2} \quad (4.39)$$

On the other hand, for any measurable subset  $E \subset \Omega$ , we have





$$\begin{aligned}
 & \int_E |g_n(x, u_n, \nabla u_n)| dx + \int_E |u_n|^{p_0-1} dx \\
 & \leq b(h(\eta)) \int_E \left( c(x) + \sum_{i=1}^N |D^i T_{h(\eta)}(u_n)|^{p_i} \right) dx \\
 & + \int_E |T_{h(\eta)}(u_n)|^{p_0-1} dx + \int_{\{h(\eta) \leq |u_n|\}} |g_n(x, u_n, \nabla u_n)| dx \\
 & + \int_{\{h(\eta) \leq |u_n|\}} |u_n|^{p_0-1} dx
 \end{aligned} \tag{4.40}$$

From (4.37), there exists  $\beta(\eta) > 0$  such that, for all  $E \subseteq \Omega$ , with  $meas(E) \leq \beta(\eta)$ , we have

$$b(h(\eta)) \int_E \left( c(x) + \sum_{i=1}^N |D^i T_{h(\eta)}(u_n)|^{p_i} \right) dx + \int_E |T_{h(\eta)}(u_n)|^{p_0-1} dx \leq \frac{\eta}{2} \tag{4.41}$$

Finally, by combining (4.39), (4.40) and (4.41), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)| dx + \int_E |u_n|^{p_0-1} dx \leq \eta \tag{4.42}$$

for all  $E$  such that  $meas(E) \leq \beta(\eta)$ ,

we then deduce that  $(g_n(x, u_n, \nabla u_n))_n$  and  $(|u_n|^{p_0-2} u_n)_n$  are equi-integrable, and by Vitali's Theorem we deduce that

$$g(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ and } |u_n|^{p_0-2} u_n \rightarrow |u|^{p_0-2} u \text{ in } L^1(\Omega) \tag{4.43}$$

#### Step 5: Passing to the limit.

Let  $v \in K_\psi \cap L^\infty(\Omega)$ , by taking  $v = u_n - \eta T_k(u_n - \varphi)$  as a test function in (4.2), with  $\eta$  small enough and  $\varphi \in K_\psi \cap W_0^{1,\vec{p}}(\Omega)$ , and putting  $M = k + \|\varphi\|_\infty$ , we get obtain

$$\begin{aligned}
 & \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\
 & + \int_\Omega |u_n|^{p_0-2} u_n T_k(u_n - \varphi) dx \leq \int_\Omega f_n T_k(u_n - \varphi) dx
 \end{aligned} \tag{4.44}$$

On one hand, if  $|u_n| > M$  then  $|u_n - \varphi| \geq |u_n| - |\varphi| > k$ , therefore  $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$ , which implies that



$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx \\
 & + \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & = \sum_{i=1}^N \int_{\Omega} (a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla \varphi)) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & + \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla \varphi) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \tag{4.45}
 \end{aligned}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx \\
 & \geq \sum_{i=1}^N \int_{\Omega} (a_i(x, T_M(u_n), \nabla T_M(u_n)) \\
 & \quad - a_i(x, T_M(u_n), \nabla \varphi)) (D^i T_M(u_n) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & + \lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla \varphi) (D^i T_M(u_n) \\
 & \quad - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \tag{4.46}
 \end{aligned}$$

The second term in the right hand side of (4.46) is equal to

$$\sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla \varphi) (D^i T_M(u) - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx$$

Therefore, we get

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) dx \\
 & \geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u), \nabla \varphi) (D^i T_M(u) - D^i \varphi) \chi_{\{|u - \varphi| \leq k\}} dx \\
 & = \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) (D^i u - D^i \varphi) \chi_{\{|u_n - \varphi| \leq k\}} dx \\
 & = \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) D^i T_k(u - \varphi) dx
 \end{aligned}$$

On the other hand, we have  $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$  weak-\* in  $L^\infty(\Omega)$ , and in view of (4.43), We obtain,





$$\int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \rightarrow \int_{\Omega} g_n(x, u, \nabla u) T_k(u - \varphi) dx, \quad (4.47)$$

$$\int_{\Omega} |u_n|^{p_0-2} u_n T_k(u_n - \varphi) dx \rightarrow \int_{\Omega} |u|^{p_0-2} u T_k(u - \varphi) dx \quad (4.48)$$

$$\text{and } \int_{\Omega} f_n T_k(u_n - \varphi) dx \rightarrow \int_{\Omega} f T_k(u - \varphi) dx \quad (4.49)$$

Again, since  $T_k(u_n - \varphi) \rightarrow T_k(u - \varphi)$  in  $W_0^{1,\bar{p}}(\Omega)$ . Which completes the proof of Theorem 4.1. for any positive function  $v \in W_0^{1,\bar{p}}(\Omega) \cap L^\infty(\Omega)$ .

## 5- Appendix

**Lemma 5.1** The bounded operator  $B_n = A_n + G_n$  acted from  $W_0^{1,\bar{p}}(\Omega)$  into  $W^{-1,\bar{p}'}(\Omega)$  is pseudo-monotone. Moreover,  $B_n$  is coercive in the following sense : there exists  $v_0 \in K_\psi$  such that

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1,\bar{p}}} \rightarrow \infty \quad \text{as } \|v\|_{1,\bar{p}} \rightarrow \infty \quad \text{for } v \in K_\psi.$$

### Proof of Lemma 5.1

Using the *Holdér's* inequality and the growth condition (3.1), we can show that the operator  $A_n$  is bounded and by (5.2) we conclude that  $B_n$  is bounded. For the coercivity, let  $v_0 \in K_\psi$ , for any  $v \in K_\psi$  we have

$$\begin{aligned} \langle A_n v, v_0 \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(v), \nabla v) D^i v_0 dx + \int_{\Omega} |v|^{p_0-2} v v_0 dx \\ &\leq \sum_{i=1}^N \left( \int_{\Omega} |a_i(x, T_n(v), \nabla v)|^{p'_i} dx \right)^{\frac{1}{p'_i}} \|D^i v_0\|_{p_i} \\ &\quad + \left( \int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \|v_0\|_{p_0} \\ &\leq C_{15} \sum_{i=1}^N \left( \int_{\Omega} (R_i(x)^{p'_i} + n^{(p-1)p'_i} + |D^i v|^{p_i}) dx \right)^{\frac{1}{p'_i}} \|v_0\|_{1,\bar{p}} \\ &\quad + \left( \int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \|v_0\|_{1,\bar{p}} \end{aligned}$$

and

$$\langle A_n v, v_0 \rangle = \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(v), \nabla v) D^i v dx + \int_{\Omega} |v|^{p_0} dx \geq \alpha' \sum_{i=1}^N \int_{\Omega} |D^i v|^{p_i} dx$$

with  $\alpha' = \min \alpha$

It follows that





$$\begin{aligned} \frac{\langle A_n v, v - v_0 \rangle}{\|v\|_{1, \bar{p}}} &= \frac{\alpha'}{\|v\|_{1, \bar{p}}} \sum_{i=1}^N \int_{\Omega} |D^i v|^{p_i} dx - \frac{\|v_0\|_{1, \bar{p}}}{\|v\|_{1, \bar{p}}} \left( \int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \\ &+ C_{15} \frac{\|v_0\|_{1, \bar{p}}}{\|v\|_{1, \bar{p}}} \sum_{i=1}^N \left( \int_{\Omega} (R_i(x)^{p_i'} + n^{(p-1)p_i'} + |D^i v|^{p_i})^{p_i'} \right)^{\frac{1}{p_i'}} \|v_0\|_{1, \bar{p}} \\ &+ \frac{\|v_0\|_{1, \bar{p}}}{\|v\|_{1, \bar{p}}} \left( \int_{\Omega} |v|^{p_0} dx \right)^{\frac{1}{p_0}} \end{aligned} \quad (5.1)$$

Using the *Holdër's* type inequality, we have for all  $u, v \in W_0^{1, \bar{p}}(\Omega)$ ,

$$\begin{aligned} \langle G_n v, v \rangle &= \int_{\Omega} g_n(x, v, \nabla v) v dx \leq \|g_n(x, v, \nabla v)\|_{p_i'} \|v\|_{p_i} \\ &\leq \left( \int_{\Omega} |g_n(x, v, \nabla v)|^{p_i'} dx + 1 \right)^{\frac{1}{p_i'}} \|v\|_{1, \bar{p}} \\ &\leq (n^{p_i'} \text{meas}(\Omega) + 1)^{\frac{1}{p_i'}} \|v\|_{1, \bar{p}} \leq C_{16} \|v\|_{1, \bar{p}} \end{aligned} \quad (5.2)$$

$$\begin{aligned} \langle G_n v, v_0 \rangle &= \int_{\Omega} g_n(x, v, \nabla v) v_0 dx \\ &\leq \left( \int_{\Omega} |g(x, v, \nabla v)|^{p_i'} dx \right)^{\frac{1}{p_i'}} \|v_0\|_{p_i} \\ &\leq C_{17} \sum_{i=1}^N \left( \int_{\Omega} (c(x)^{p_i'} + |D^i v|^{p_i})^{p_i'} dx \right)^{\frac{1}{p_i'}} \|v_0\|_{1, \bar{p}} \end{aligned} \quad (5.3)$$

It follows that

$$\frac{\langle G_n v, v - v_0 \rangle}{\|v\|_{1, \bar{p}}} = C_{16} \frac{\|v_0\|_{1, \bar{p}}}{\|v\|_{1, \bar{p}}} - C_{17} \frac{\|v_0\|_{1, \bar{p}}}{\|v\|_{1, \bar{p}}} \sum_{i=1}^N \left( \int_{\Omega} (c(x)^{p_i'} + |D^i v|^{p_i})^{p_i'} dx \right)^{\frac{1}{p_i'}} \quad (5.4)$$

From (5.1) and (5.4), as  $\|v\|_{1, \bar{p}} \rightarrow \infty$ , we conclude that

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1, \bar{p}}} = \frac{\langle A_n v, v - v_0 \rangle}{\|v\|_{1, \bar{p}}} + \frac{\langle G_n v, v - v_0 \rangle}{\|v\|_{1, \bar{p}}} \rightarrow \infty \quad (5.5)$$

It remains to show that  $B_n$  is pseudo-monotone. Let  $(u_k)_{k \in \mathbb{N}}$  be a sequence in  $W_0^{1, \bar{p}}(\Omega)$  such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W^{1, \bar{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{1, \bar{p}'}(\Omega) \\ \limsup_{k \rightarrow \infty} \langle B_n u_k, u_k \rangle \leq \langle \chi_n, u \rangle \end{cases} \quad (5.6)$$

We will prove that  $\chi_n = B_n u$  and  $\langle B_n u_k, u_k \rangle \rightarrow \langle \chi_n, u \rangle$  as  $k \rightarrow \infty$

Firstly, in view of the compact embedding  $W^{1, \bar{p}}(\Omega) \hookrightarrow L^p(\Omega)$  we have  $u_k \rightarrow u$  in  $L^p(\Omega)$  for a subsequence denoted again  $(u_k)_{k \in \mathbb{N}}$ .





As  $(u_k)_{k \in \mathbb{N}}$  is a bounded sequence in  $W^{1,\vec{p}}(\Omega)$ , then by the growth condition  $(a_i(x, T_n(u_k), \nabla u_k))_{k \in \mathbb{N}}$  is bounded in  $L^{p'_i}(\Omega)$ . Therefore, there exists a function  $\varphi_i \in L^{p'_i}(\Omega)$  such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{in} \quad L^{p'_i}(\Omega) \quad \text{as} \quad k \rightarrow \infty \quad (5.7)$$

and we have

$$|u_k|^{p_0-2} u_k \rightarrow |u|^{p_0-2} u \quad \text{in} \quad L^{p'_0}(\Omega) \quad \text{as} \quad k \rightarrow \infty \quad (5.8)$$

On the one hand, we have for all  $v \in W_0^{1,\vec{p}}(\Omega)$

$$\langle B_n u_k, u_k \rangle \rightarrow \langle \chi_n, u \rangle \quad \text{as} \quad k \rightarrow \infty$$

$$\begin{aligned} \langle \chi_n, v \rangle &= \lim_{k \rightarrow \infty} \langle B_n u_k, u_k \rangle \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i v \, dx + \lim_{k \rightarrow \infty} \int_{\Omega} |u_k|^{p_0-2} u_k v \, dx \\ &= \sum_{i=1}^N \int_{\Omega} \varphi_i D^i v \, dx + \int_{\Omega} |u|^{p_0-2} u v \, dx \end{aligned} \quad (5.9)$$

From relations (5.6) and (5.9), we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle B_n u_k, u_k \rangle &= \limsup_{k \rightarrow \infty} \left( \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_{\Omega} |u_k|^{p_0} \, dx \right) \\ &\leq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u \, dx + \int_{\Omega} |u|^{p_0} \, dx. \end{aligned} \quad (5.10)$$

On the other hand, by (3.3) we get

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u)) (D^i u_k - D^i u) \, dx \\ + \lim_{k \rightarrow \infty} \int_{\Omega} (|u_k|^{p_0-2} u_k - |u|^{p_0-2} u) (u_k - u) \, dx \geq 0 \end{aligned} \quad (5.11)$$

Then

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \\ + \int_{\Omega} |u_k|^{p_0} \, dx \geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u \, dx \\ + \int_{\Omega} |u_k|^{p_0-2} u_k u \, dx + \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u) (D^i u_k - D^i u) \, dx \\ + \int_{\Omega} |u|^{p_0-2} u (u_k - u) \, dx \end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have





$T_n(u_k) \rightarrow T_n(u)$  in  $L^{p_i}(\Omega)$  then  $a_i(x, T_n(u_k), \nabla u) \rightarrow a_i(x, T_n(u), \nabla u)$  in  $L^{p_i}(\Omega)$  and by (5.7), we get

$$\liminf_{k \rightarrow \infty} \left( \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx + \int_{\Omega} |u_k|^{p_0} dx \right) \geq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u dx + \int_{\Omega} |u|^{p_0} dx, \quad (5.12)$$

and

$$\int_{\Omega} g_n(x, u_k, \nabla u_k) u_k dx \rightarrow \int_{\Omega} \phi_i u dx \quad (5.13)$$

Having in mind (5.10) and (5.12) we conclude that

$$\lim_{k \rightarrow \infty} \left( \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx + \int_{\Omega} |u_k|^{p_0} dx \right) = \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u dx + \int_{\Omega} |u|^{p_0} dx, \quad (5.14)$$

According to (5.8), (5.13) and (5.14), we obtain

$$\langle B_n u_k, u_k \rangle \rightarrow \langle \chi_n, u \rangle \text{ as } k \rightarrow \infty$$

Now, by (5.14) we can prove that

$$\lim_{k \rightarrow \infty} \left( \sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u_k)) (D^i u_k - D^i u) dx + \int_{\Omega} (|u_k|^{p_0-2} u_k - |u|^{p_0-2} u)(u_k - u) dx \right) = 0$$

So, by virtue of **Lemma 3.2**, we get

$$u_k \rightarrow u \text{ in } W_0^{1, \vec{p}}(\Omega) \text{ and } D^i u_k \rightarrow D^i u \text{ a.e. in } \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightarrow a_i(x, T_n(u), \nabla u) \text{ in } L^{p'_i}(\Omega) \text{ for } i = 1, \dots, N,$$

and

$$g_n(x, u_k, \nabla u_k) \rightarrow g_n(x, u, \nabla u) \text{ in } L^{p'_0}(\Omega),$$

which implies that  $\chi = B_n u$ , which completes the proof of **lemma 5.1**.

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